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# On Completeness of Eigenfunctions of the One-Speed Transport Equation

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**Abstract.** It is shown that the set of Case's eigenfunctions of the one speed transport equation is complete in the rigged Hilbert space $W_2^1([-1,1]) \subset L_2(-1,1) \subset W_2^{-1}([-1,1])$.

## 1. Introduction

Case's method of singular eigenfunction expansions for solving the transport equation [1] seeks, by separation of variables, to construct a sufficiently rich set of solutions, called elementary solutions, which would enable one to expand an "arbitrary" solution of the equation into a Fourier series in terms of this set. An important point in this method is the completeness proof for the set of elementary solutions. Originally it was shown [2], by means of the theory of singular integral equations, that the expansion coefficients are uniquely determined for the class of Hölder-continuous functions, which, within this class, proves the completeness. An alternative to this constructive approach is the demonstration of the closure relation for the set of elementary solutions [1]. Unfortunately, either proof has to be carried out separately for each particular from of the transport equation under consideration, and moreover, there remains some doubt as to whether the obtained result is the strongest possible.

According to an idea by A. Skumanich, commented upon in Ref. [1], the completeness proof for Case's elementary solutions should be based on more general arguments, provided by the functional-analytic properties of the underlaying transport operator. This would lead to the completeness proof for a whole class of operators which have certain common properties.

The functional analytic approach to the problem was considered by Hangelbroek [3] and by Larsen and Habetler [4], where it was essentially shown that the Case eigenfunction expansion formula represents the resolution of the identity of a transport operator, but again only after resorting to a kind of Hölder continuity requirement. There remains some ambiguity about the notion of the eigenfunction, which is also referred to by Baird and Zweifel [5], and the structure of the space of eigenfunctions remains unclear.

Here we propose a completeness proof which is based on the theory of eigenfunction expansions for self adjoint operators in rigged Hilbert spaces, as expounded in the treatise by Berezanskiĭ [6]. A rigged Hilbert space, of the type to be considered, is a triple of separable Hilbert spaces $H_+ \subset H \subset H_-$, where $H_+$, the positive space, is dense in $H$, and $H_-$, the negative space, is isometric to the