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A Simple Proof of the GHS and Further Inequalities

Richard S. Ellis*
Department of Mathematics, Northwestern University, Evanston, Illinois, USA

James L. Monroe**
Department of Physics, Northwestern University, Evanston, Illinois, USA

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Abstract. We formulate and prove a general set of correlation inequalities for spin $-1/2$ Ising ferromagnets with pair interactions. One of these is the Griffiths-Hurst-Sherman inequality. The proof is obtained using Gaussian random variables.

1. Introduction

We consider a system of N Ising spins with ferromagnetic pair interactions and non-negative external magnetic field. The probability $\mu(\sigma)$ of any configuration $\sigma=(\sigma_1,\ldots,\sigma_N)$, $\sigma_i=\pm 1$, is given by the formula $\mu(\sigma)=Z^{-1}\exp(-\beta H(\sigma))$, where $\beta=(kT)^{-1}$,

$$H(\sigma)=-\frac{1}{2}\sum_{i\neq j}J_{ij}\sigma_i\sigma_j-h\sum_i\sigma_i\,,\qquad J_{ij}=J_{ji}\geqq 0\,,\qquad h\geqq 0\,,\tag{1.1}$$

$$Z=\sum_{\{\sigma\}}\exp(-\beta H(\sigma))\,.\tag{1.2}$$

In the sequel, we set $\beta=1$. Given spin sites i,j,k, we define the third Ursell function

$$\begin{aligned}u_3(i,j,k)\equiv\langle\sigma_i\sigma_j\sigma_k\rangle-\langle\sigma_i\rangle\langle\sigma_j\sigma_k\rangle-\langle\sigma_j\rangle\langle\sigma_i\sigma_k\rangle\\-\langle\sigma_k\rangle\langle\sigma_i\sigma_j\rangle+2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle\,,\end{aligned}\tag{1.3}$$

where the bracket $\langle\,\rangle$ denotes the expected value with respect to the measure μ.

The Griffiths-Hurst-Sherman inequality (hereafter GHS inequality) states that

$$u_3(i,j,k)\leqq 0\,.\tag{1.4}$$

An important consequence of this inequality is that the average magnetization per site is a concave function of magnetic field h, a fact needed for the proof of certain critical point exponent inequalities [1]. It has also been used by Preston [2] to show the absence of phase transitions in the thermodynamic limit for $h\neq 0$.

Inequality (1.4) was first proved by Griffiths, Hurst, and Sherman [1] and later by Lebowitz [3]. Our proof is completely self-contained and, we believe, is much simpler. It is based on ideas introduced by Monroe and Siegert [4], who obtained simple proofs of the GKS inequalities [5]. Similar methods have also been used by Monroe [6] to prove certain FKG inequalities [7]. At the end of the next section, we mention additional new inequalities which are proved by the same technique.

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