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Correlation Inequalities and the Mass Gap in $P(\phi)_2$

III. Mass Gap for a Class of Strongly Coupled Theories with Nonzero External Field

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Abstract. We consider the infinite volume Dirichlet (or half-Dirichlet) $P(\phi)_2$ quantum field theory with $P(X)=aX^4+bX^4+bX^2-\mu X\,(a>0)$. If $\mu\neq 0$ there is a positive mass gap in the energy spectrum. If the gap vanishes as $\mu\to 0$, it goes to zero no faster than linearly yielding a bound on a critical exponent.

§ 1. Introduction

In this paper, we discuss various aspects of the $P(\phi)_2$ Euclidean field theory [28, 23]. In the statistical mechanical approach to these theories which we have advocated elsewhere [10] (see also our contributions to [28]), one of the subprograms concerns the use of Ising model techniques. These techniques are especially useful in the study of the $:a\phi^4+b\phi^2-\mu\phi:_2$ theory where both the lattice approximation [10] and classical Ising approximation [24] are available. In fact, in II of this series [21], we used these techniques to complete the proof of the Wightman axioms for these theories when $\mu\neq 0$. In essence, the result of that note was that 0 was a simple eigenvalue of the Hamiltonian in the infinite volume Dirichlet theory. Using very different techniques, based in part on the cluster expansion of [7, 8], Spencer [25] proved that the theories with $|\mu|$ large (and periodic B.C.) have a mass gap, i.e. that 0 is a simple, *isolated*, eigenvalue of the Hamiltonian. Our goal in this note is to extend this result to any $\mu\neq 0$.

As before, our proof is modelled on a result in the theory of Ising models, namely the recent work of Lebowitz and Penrose [14, 15] on clustering. They, in turn, rely on subharmonicity ideas first introduced by Penrose and Elvey [16]. In the present context, this basic idea of "superharmonic continuation" is very simple and beautiful: Let $m_l(\mu)$ be the mass gap for the (periodic) Hamiltonian on $[-l/2, l/2]$ with interaction polynomial $P(X)=aX^4=bX^2-\mu X$. We show that $m_l(\mu)$ has a continuation to a *nonnegative superharmonic* function $M_l(\mu)$ in the region $\mathrm{Re}\,\mu>0$ where the Lee-Yang theorem of the classical Ising approximation applies [24]. Now for large real μ, Spencer [25] assures us that $M_l(\mu)$ is bounded

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