Commun. math. Phys. 40, 215—222 (1975)

On the Nonrelativistic Limit of the Dirac Theory

W. Hunziker

Institut für Theoretische Physik der ETH, Zürich, Switzerland

Received October 15, 1974

Abstract. The relation between a "nonrelativistic" Hamiltonian of the form $H^\infty = (A+B)^2 + C$ and a corresponding family of "Dirac-Hamiltonians" $H(c)$ in the limit $c \to \infty$ is investigated. It is shown that the resolvent $(z - H(c))^{-1}$ and the relativistic perturbation of isolated eigenvalues of H^∞ are analytic in $1/c$ for sufficiently large $|c|$.

1. Introduction

The Hamiltonian of a Dirac-electron of charge $e=1$ and mass $m=1/2$ may be written as

$$H(c) = c\boldsymbol{\alpha}(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})) + \tfrac{1}{2}\beta c^2 + \varphi(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{p} = -id/d\boldsymbol{x}$ and with the 4×4-matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

whose elements are the 2×2-matrices $\mathbb{1}$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) =$ set of Pauli spin-matrices. $\boldsymbol{A}(\boldsymbol{x})$ and $\varphi(\boldsymbol{x})$ are the potentials of the static electromagnetic field. The usual factor $1/c$ in front of $\boldsymbol{A}(\boldsymbol{x})$ is omitted on purpose since it must be kept fixed in the nonrelativistic limit $c \to \infty$. $H(c)$ acts on the Hilbertspace $C^4 \otimes L^2(R^3)$ of square-integrable 4-component wave functions.

On a formal level, it is well understood that the nonrelativistic limit $c \to \infty$ is described by the Pauli-Hamiltonian

$$H^\infty = (\boldsymbol{\sigma}(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))^2) + \varphi(\boldsymbol{x}) \tag{2}$$

on the smaller Hilbertspace $C^2 \otimes L^2(R^3)$, and there exists a sytematic scheme for obtaining corrections to H^∞ in the form of a power series in $1/c$ [1]. However, these "relativistic perturbations" of H^∞ are given by more and more singular operators which are by no means small with respect to H^∞. One might therefore suspect that perturbation expansions in powers of $1/c$ are at best asymptotic.

Nevertheless, Titchmarsh [2] has proved analyticity in $1/c$ of eigenvalues and eigenfunctions for the spherically symmetric case without magnetic field: $\varphi = \varphi(r)$, $\boldsymbol{A} = 0$; and Veselić [3] has extended this result to the case without spherical symmetry: $\varphi = \varphi(\boldsymbol{x})$, $\boldsymbol{A} = 0$.

In this note we investigate the general case $\boldsymbol{A} \neq 0$ which poses essentially new problems-already in the nonrelativistic limit. One of the points we wish to make is that it is profitable to treat a general Hamiltonian of type $H^\infty = (A+B)^2 + \mathrm{C}$ as a nonrelativistic limit of a corresponding Dirac-Hamiltonian $H(c)$.