Commun. math. Phys. 40, 125—146 (1975)

An Example in Potential Scattering Illustrating the Breakdown of Asymptotic Completeness

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Received August 16, 1974

Abstract. An example is given of a local spherically symmetric short range potential, such that the wave operators for scattering of a single particle by the potential are not complete. States exist which are asymptotically free at $t=-\infty$, but having a non zero probability of absorption into the origin at $t=+\infty$.

1. Introduction

It is a somewhat unusual situation that, although asymptotic completeness in potential scattering has received considerable attention, and has been proved under a wide variety of rather weak assumptions, there does not seem to exist in the literature any example of a short range local potential for which completeness of the wave operators does *not* hold.

$V(\boldsymbol{r})$ is said to be of short range if $|V(\boldsymbol{r})| = O(r^{-(1+\varepsilon)})$ as $r \to \infty (\varepsilon > 0)$. If in addition $V(\boldsymbol{r})$ is locally square integrable in $\mathbb{R}^3 \setminus \{0\}$ the wave operators

$\Omega_{\pm} = \underset{t \to \mp\infty}{s\text{-lim}}\, e^{iHt} e^{-iH_0 t}$ exist [1], and are said to be complete if range $(\Omega_+) =$ range (Ω_-), or equivalently if the scattering operator $S = \Omega_-^* \Omega_+$ is unitary [2, Chapter IV].

With an attractive singular spherical potential such as $-r^{-n}$ $(n \geqq 2)$ one has in the classical theory the phenomenon of the particle plunging into the origin (within a finite time interval). The expectation that this might occur also in the quantum-mechanical case and lead to a breakdown of completeness turns out not to be justified. Since the total Hamiltonian is not essentially self-adjoint for such a potential, it is possible to define an evolution by means of a non-unitary semi-group, of which the generator is not self-adjoint (see [3, 4] for the case $n=2$, and also [5] which relates such a semi-group to the one-parameter family of unitary [6] evolutions); however, completeness holds for all possible *unitary* evolutions [7, 8].

Nor is completeness violated for singular repulsive potentials [9, 10], or for potentials which are non-singular [11] (i.e. for which $\int_0^1 r|V(|\boldsymbol{r}|)| \cdot dr < \infty$). Hence it remains only to consider potentials which are both unbounded and oscillating near $r=0$. (Potentials which are unbounded

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