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A Generalization of the FKG Inequalities

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Abstract. We generalize a theorem of Holley to include the case of continuous spins. Holley's theorem is itself a generalization of the inequalities due to Fortuin, Kastelyn and Ginibre.

1. Introduction

In the study of correlation functions for the Ising and other lattice models in statistical mechanics the inequalities of Fortuin, Kastelyn and Ginibre [2] (the FKG inequalities) play a fundamental role. The object of this paper is to give a proof of some generalized FKG inequalities which include the case of continuous spins. Results of this type have been obtained from the original FKG inequalities by using discrete approximations (see [5]); also a direct proof has been given by Cartier [6]. In this paper we will in fact generalize a result of Holley [3], which easily implies the FKG inequalities. Let Λ be a finite set and let $\mathscr{P}(\Lambda)$ denote the set of subsets of Λ. Suppose μ_1, $\mu_2 : \mathscr{P}(\Lambda) \to \mathbb{R}$ are probability densities, i.e. $\mu_i \geqq 0$ and

$$\sum_{A \subset \Lambda} \mu_i(A) = 1 \quad \text{for} \quad i = 1, 2 .$$

Then we have:

Theorem 1 (Holley [3]). *If for all* $A, B \in \mathscr{P}(\Lambda)$

$$\mu_1(A \cup B)\, \mu_2(A \cap B) \geqq \mu_1(A)\, \mu_2(B)$$

then

$$\sum_{A \subset \Lambda} h(A)\, \mu_1(A) \geqq \sum_{A \subset \Lambda} h(A)\, \mu_2(A)$$

for any increasing $h : \mathscr{P}(\Lambda) \to \mathbb{R}$ *(where by increasing we mean that* $h(A) \geqq h(B)$ *whenever* $A \supset B$*).*

Using the well-known result of Birkhoff [1] that any finite distributive lattice is isomorphic to some sub-lattice of $\mathscr{P}(\Lambda)$ for some finite set Λ, it follows that Theorem 1 is true for any finite distributive lattice (where we replace $\cup$ by $\vee$ and $\cap$ by $\wedge$). From Theorem 1 we get the FKG inequalities.