Commun. math. Phys. 34, 193—222 (1973)

Additivity of the Entropy and Definition of the Temperature for Quantum Systems

H. Roos

Institute for Theoretical Physics, Groningen, The Netherlands*

Received April 12; in revised form July 27, 1973

Abstract. A closed quantum system $\mathscr{S}$ is considered which is described by a microcanonical ensemble. $\mathscr{S}$ consists of two rather weakly interacting subsystems $\mathscr{S}_1, \mathscr{S}_2$. In a rigorous way, the additivity of the entropy is proved by deriving an expression for the entropy density of $\mathscr{S}$ in terms of the entropy densities of $\mathscr{S}_1$ and $\mathscr{S}_2$. "Rigorous" implies that the thermodynamic limit is taken. In the second part, it is shown how a microcanonical state $\omega_\varepsilon(A) = \lim_{\Lambda\to\infty} \frac{\operatorname{Tr}\delta^\Lambda(H(\Lambda)-E)A}{\operatorname{Tr}\delta^\Lambda(H(\Lambda)-E)}$ of the composite system – provided this limit exists – gives rise to a "canonical" state ω^β, when restricted to $\mathscr{S}_1$, provided $\mathscr{S}_1$ is very "small" as compared to $\mathscr{S}_2$; ω^β is defined as a limit of Gibbs states. This yields a definition of the equilibrium temperature β^{-1}.

I. Introduction

The concept of temperature can be introduced in statistical mechanics in several ways. Up to now there does not exist a general proof of the equality of time averages and ensemble averages of observables; it has to be taken as an axiom that there are suitable ensembles. One can start with canonical or grand canonical ensembles with partition functions depending on a parameter β and show that everything works if β is the inverse temperature. A more satisfactory way of introducing the temperature is to consider several microcanonical ensembles representing closed systems, provide them with a – however weak – thermal interaction and show that there is a parameter govering the equilibrium between them, which is to be defined as the temperature (or a function thereof). A third possibility is, to deal with a "large" system in thermal contact with a "small" system and show that the "small" one can be described by a canonical ensemble with a parameter β depending only on the "large" system, the heat reservoir. This is well known; in this paper, we aim at giving new proofs considering quantum systems from the very beginning and working in the context of rigorous statistical mechanics as developed by Ruelle and many others.

* On leave of absence from the Institut für Theoretische Physik, Universität Göttingen.