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Golden-Thompson and Peierls-Bogolubov Inequalities for a General von Neumann Algebra

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Abstract. Some inequalities for a general von Neumann algebra, which reduces to Golden-Thompson and Peierls-Bogolubov inequalities when the von Neumann algebra has a trace, are proved.

§ 1. Main Results

Golden-Thompson and Peierls-Bogolubov inequalities are extended to von Neumann algebras, which have traces, by Ruskai [5]. We shall extend them to a general von Neumann algebra. Because a von Neumann algebra does not necessarily have a trace, we use the notion of relative Hamiltonian [3] instead of the trace.

Let $\mathfrak{M}$ be a von Neumann algebra and Ψ be a cyclic and separating vector. Let $\psi(x) = (x\Psi, \Psi)$ for $x \in \mathfrak{M}$. For a self-adjoint h in $\mathfrak{M}$, a vector $\Psi(h)$ is defined by

$$\Psi(h) \equiv \sum_{n=0}^{\infty} \int_0^{1/2} \mathrm{d}t_1 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \, \Delta_\Psi^{t_n} h \, \Delta_\Psi^{t_{n-1}-t_n} h \ldots \Delta_\Psi^{t_1-t_2} h \Psi \,, \tag{1.1}$$

where Δ_Ψ is the modular operator for Ψ. (As we shall see in (3.6), it is also possible to write $\Psi(h) = e^{(H+h)/2}\Psi$ where $H = \log \Delta_\Psi$.)

Theorem 1. *If* $\|\Psi\| = 1$, *then*

$$\|\Psi(h)\|^2 \geqq \exp \psi(h) \,. \tag{1.2}$$

Theorem 2.

$$\psi(e^h) \geqq \|\Psi(h)\|^2 \,. \tag{1.3}$$

The connection with Golden-Thompson and Peierls-Bogolubov inequalities for finite matrices can be seen as follows.

Let $\mathfrak{M}$ be a finite matrix algebra and Ω be a cyclic and separating vector such that $(x\Omega, \Omega) = \operatorname{tr} x$ for $x \in \mathfrak{M}$. Let $\Psi = (\operatorname{tr} e^A)^{-1/2} e^{A/2}\Omega$ for a self-adjoint A in $\mathfrak{M}$. Then Ψ is a unit cyclic and separating vector.