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Remarks on Two Theorems of E. Lieb

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Abstract. The concavity of two functions of a positive matrix A, $\operatorname{Tr}\exp(B+\log A)$ and $\operatorname{Tr} A^r K A^p K^*$ (where $B=B^*$ and K are fixed matrices), recently proved by Lieb, can also be obtained by using the theory of Herglotz functions.

In a recent article [1], Lieb has shown, among other things, that, if A_1, A_2, B, K are complex matrices, with $A_1 = A_1^*$, $A_2 = A_2^* > 0$, $B = B^*$, the two functions $t \to \operatorname{Tr}\exp(B + \log(tA_1 + A_2))$ $t \to \operatorname{Tr}(tA_1 + A_2)^r K \cdot (tA_1 + A_2)^p K^*$ (where $0 < r, 0 < p, r + p = s \leqq 1$), are concave functions of the real variable t for sufficient by small t. The object of this note is to indicate how this can also be seen by using the theory of Herglotz functions: in fact, for $A_1 > 0$, the two above mentioned functions can be extended to Herglotz functions holomorphic in the complex plane cut along the real axis from $-\infty$ to $\tau \geqq 0$. Some supplementary work is necessary to study the case of arbitrary self-adjoint A_1. The applicability of the method obviously extends beyond the examples treated here.

Note. in this paper, if A is an element of a C^*-algebra $\mathscr{A}$ with unit, we write $A \geqq 0$ to mean $A = B^* B$ for some $B \in \mathscr{A}$, and $A > 0$ to mean that, for some real number $a > 0$, the inequality $A - a \quad \geqq 0$ holds. Of course $A > 0$ is equivalent to: $A \geqq 0$ and A^{-1} exists as an element of $\mathscr{A}$.

I. Remarks

Let $\mathscr{A}$ be a C^* algebra with unit.

1. Let $A \in \mathscr{A}$ and let $\operatorname{Sp} A$ denote its spectrum. Suppose f is a complex function holomorphic in an open set of the complex plane containing $\operatorname{Sp} A$. Then $f(A)$ can be defined (as a holomorphic function of A with values in $\mathscr{A}$) by

$$f(A) = \frac{1}{2\pi i} \int_{\mathscr{C}} f(z)\,(z - A)^{-1}\, dz$$

where $\mathscr{C}$ is a contour surrounding $\operatorname{Sp} A$. All reasonable definitions of $f(A)$ coincide with this and:

$$\operatorname{Sp} f(A) \subset f(\operatorname{Sp} A)$$

(see [2], Chapter I, § 4, Proposition 8, p. 47).