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Some Remarks on the Location of Zeroes of the Partition Function for Lattice Systems

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Abstract. We use techniques which generalize the Lee-Yang circle theorem to investigate the distribution of zeroes of the partition function for various classes of classical lattice systems.

0. Introduction

The Lee-Yang circle theorem [12] remains one of the very few effective tools which are at our disposal in the rigorous theory of phase transitions. An important conceptual clarification of this theorem, as well as an extension to quantum systems were given by Asano [2]. This work was continued by Suzuki-Fisher [19]. A generalization of the Lee-Yang theorem to noncircular regions by the present author [17] also benefitted from Asano's ideas. Here some more facts concerning the position of zeroes of the partition function Z for lattice systems are presented. In particular, results due to Heilmann-Lieb [11], Heilmann [10], and Runnels-Hubbard [18] are recovered. Although there is as yet no general method for locating the zeroes of Z, the techniques known so far permit to say something in a fairly large number of cases.

The grand partition function Z for a lattice gas is a polynomial in one complex variable z (the activity). To locate the zeroes of $Z(z)$ it is convenient, as Lee and Yang already remarked, to work with a polynomial P in n variables such that

$$Z(z) = P(z, \ldots, z)$$

and to prove that $P(z_1, \ldots, z_n) \neq 0$ when the z_i are away from certain regions of the complex plane. $P(z_1, \ldots, z_n)$ is the grand partition function for a system having a different activity z_i at each lattice site, it is a polynomial of first degree in each argument separately.

Let Λ be a finite lattice subset and P the partition function for Λ with some given interaction between sites. The arguments of P are the z_x with $x \in \Lambda$ and

$$P = \sum_{X \subset \Lambda} e^{-\beta U(X)} \prod_{x \in X} z_x$$