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The Averaged Lagrangian and High-Frequency Gravitational Waves*

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Abstract. The averaged Lagrangian technique of Whitham is applied to the second variation Lagrangian for the perturbations of a general-relativistic spacetime. This gives a variational principle for (sums of) approximately periodic gravitational waves which in turn leads to the rederivation of some results of Isaacson. Examples of the use of the method are discussed.

1. Introduction: The Averaged Lagrangian Technique

The purpose of this paper is to discuss approximate solutions of the Einstein equations which are approximately periodic (and can be interpreted as containing high-frequency gravitational waves) by the "averaged Lagrangian" method introduced by Whitham [1]. Whitham showed that this method is closely related to the so-called "two-timing" method, which has been used for the gravity-wave problem by Choquet-Bruhat [2] and Madore [3]. In this introduction we will review these techniques, in Section 2 we discuss gravitational waves and the Lagrangian for them, and in Section 3 we apply the averaging technique to this Lagrangian. Some examples are discussed in Section 4.

The two-timing method consists of assuming that changes in the dependent variables, ψ^A say, of a problem occur on two scales; for example, a wave train may show rapid oscillation and a slow change in amplitude, frequency and wave number. One writes

$$\psi^A = \psi^A(X^\nu, \theta) \tag{1.1}$$

where $X^\mu = \varepsilon x^\mu$ and $\theta = \varepsilon^{-1}\,\Theta(X^\mu)$ and it is assumed that the derivatives of ψ^A with respect to X^μ and θ are of equal magnitude (which we may take as order unity). The small parameter ε then measures the ratio of the fast length scale to the slow one. It should be noted that rapid variations only occur in the direction of the vector

$$l_\mu = \frac{\partial\theta}{\partial x^\mu} = \frac{\partial\Theta}{\partial X^\mu} = \Theta_{,\mu}\,. \tag{1.2}$$

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