Commun. math. Phys. 29, 249—264 (1973)

Convex Structures and Operational Quantum Mechanics

S. Gudder

Department of Mathematics, University of Denver, Denver, Colorado, USA

Received May 29, 1972; in revised form September 15, 1972

Abstract. A general mathematical framework called a convex structure is introduced. This framework generalizes the usual concept of a convex set in a real linear space. A metric is constructed on a convex structure and it is shown that mappings which preserve the structure are contractions. Convex structures which are isomomorphic to convex sets are characterized and for such convex structures it is shown that the metric is induced by a norm and that structure preserving mappings can be extended to bounded linear operators.

Convex structures are shown to give an axiomatization of the states of a physical system and the metric is physically motivated. We demonstrate how convex structures give a generalizing and unifying formalism for convex set and operational methods in axiomatic quantum mechanics.

1. Introduction

Until recently the main axiomatic frameworks for quantum mechanics have been the C^*-algebra approach [6, 14] and the "quantum logic" approach [4, 7, 15]. Recently a new and more general method of attack has been discussed, which might be called the "convex set" or "operational" approach. This method has been emphasized by Ludwig *et al.* [13] Gunson [5], Mielnik [9, 10], Davies and Lewis [2] and others [3, 11]. In this framework a basic role is played by the convex set of normalized states S_1 together with the geometric properties of the boundary of S_1. Although the particular terminology, physical interpretations, and certain details of these investigations may differ, they all use convex set methods.

In this paper we introduce a framework which is even more general but which we feel forms a unification of the convex set or operational approaches and at the same time provides a new and perhaps useful mathematical tool for further investigations. In this framework the only primitive axiomatic elements are the normalized states S_1 of a physical system and the only operation postulated on S_1 is that of forming mixtures. In this way the convex structure of S_1 is isolated and is, in fact, the only structure of S_1.