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The Vacuum Energy for $P(\phi)_2$: Infinite Volume Limit and Coupling Constant Dependence*

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Abstract. Let $E(\lambda g)$ be the vacuum energy for the $P(\phi)_2$ Hamiltonian with space cutoff $g(x) \geqq 0$ and coupling constant $\lambda \geqq 0$. For suitable families of cutoffs $g \to 1$, the vacuum energy per unit volume converges; i.e., $-E(\lambda g)/\int g(x)\,dx \to \alpha_\infty(\lambda)$. We obtain bounds on the λ dependence of $\alpha_\infty(\lambda)$ for large and small λ. These lead to estimates for $E(\lambda g)$ as a functional of g that permit a weakening of the standard regularity conditions for g. Typical of such estimates is the "linear lower bound", $-E(g) \leqq \text{const} \int g(x)^2\,dx$, valid for all $g \geqq 0$ provided that P is normalized so that $P(0) = 0$. Finally we show that the perturbation series for $\alpha_\infty(\lambda)$ is asymptotic to second order.

Section 0: Introduction

This paper is a continuation of our previous investigations [6, 7] on the infinite volume behavior of the vacuum in $P(\phi)_2$. We are mainly concerned with $E(\lambda g)$, the ground state energy of the $P(\phi)_2$ Hamiltonian,

$$H(g) = H_0 + \lambda \int g(x) : P(\phi(x)) : dx\,. \tag{0.1}$$

The polynomial P is semibounded and normalized, i.e.

$$P(X) = \sum_{r=1}^{2n} b_r X^r,\ b_{2n} > 0\,,$$

the coupling constant $\lambda \geqq 0$, and the cutoff $g(x) \geqq 0$. In [6, 7] we restricted our attention to sharp space cutoffs: $g = \chi_\ell$, the characteristic function of the interval $[-\ell/2, \ell/2]$. In particular it was shown that the vacuum energy per unit volume, $\alpha_\ell = -E(\chi_\ell)/\ell$, converges to a finite constant $\alpha_\infty > 0$ as $\ell \to \infty$ [6] and that this convergence is monotone [7].

In the present paper we extend the results of [6] and [7] to more general cutoffs g. In Section 2, for example, we extend the convergence of

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