Commun. math. Phys. 29, 105—111 (1973)

Two Examples Illustrating the Differences between Classical and Quantum Mechanics

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Received August 1, 1972

Abstract. Two examples are presented: The first shows that a potential $V(x)$ can be in the limit circle case at ∞ even if the classical travel time to ∞ is infinite. The second shows that $V(x)$ can be in the limit point case at ∞ even though the classical travel time to infinity is finite. The first example illustrates the reflection of quantum waves at sharp steps. The second example illustrates the tunnel effect.

In this paper we give two examples of motion on a half-line which illustrate two physical differences between classical and quantum mechanics. It is useful to study the half-line case since the necessary techniques and estimates are elementary and the complications which arise in higher dimensions are absent. We say that a potential $V(x)$ is *classically complete at* ∞ if the classical travel time to infinity is infinite for all initial conditions. We say that $V(x)$ is *quantum mechanically complete at* ∞ if the differential operator $-\frac{1}{2m}\frac{d^2}{dx^2}+V(x)$ is in the limit point case at ∞. At first glance it seems that the two notions of completeness might be the same since $-\frac{1}{2m}\frac{d^2}{dx^2}+V(x)$ is in the limit point case at ∞ if one need not specify boundary conditions at ∞. In a rough intuitive sense, this should happen if the classical travel time is infinite. But, in fact, this rough intuition is correct only if the derivatives of $V(x)$ are "small" compared to $V(x)$. We present two examples illustrating this fact. In the first example, $V(x)$ is classically incomplete at ∞ but quantum mechanically complete; in the second, $V(x)$ is classically complete at ∞ but quantum mechanically incomplete. The examples

* This research was partially supported by the National Science Foundation under Grant GP-34260.

For help with the diagrams, the authors thank Bob Johnson.