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On the Homotopical Significance of the Type of von Neumann Algebra Factors

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Abstract. The set of all projections and the set of all unitaries in a von Neumann algebra factor $\mathscr{A}$ are studied from the homotopical point of view relative to the operator norm topology.

Two projections E and F can be deformed continuously to each other if and only if $E \sim F$ and $1-E \sim 1-F$ where $\sim$ denotes the equivalence of projections in $\mathscr{A}$ in the sense of von Neumann. In other words, the relative dimension and co-dimension are a complete homotopical invariants of projections in $\mathscr{A}$ and label pathwise connected components of the set of projections.

The first homotopy group $\pi_1(\mathscr{U}(\mathscr{A}))$ of unitaries in $\mathscr{A}$ is shown to be 0 for $\mathscr{A}$ of infinite type. For type II_1 and type I_n factors, $\pi_1(\mathscr{U}(\mathscr{A}))$ are isomorphic to additive groups of reals $\boldsymbol{R}$ and integers $\boldsymbol{Z}$, respectively, in which the first homotopy group $\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A}))$ of the center of $\mathscr{U}(\mathscr{A})$ is imbedded as $\boldsymbol{Z}$ and $n\boldsymbol{Z}$, respectively.

§ 0. Introduction

In [5, 6] Glimm's classification of U.H.F. algebras is reobtained by means of the first homotopy group $\pi_1(\mathscr{U}(\mathscr{A}))$ of the unitary group $\mathscr{U}(\mathscr{A})$ of a U.H.F. C^*-algebra $\mathscr{A}$ and the canonical homomorphism $\varphi: \pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A})) \to \pi_1(\mathscr{U}(\mathscr{A}))$ where $\mathscr{Z}\mathscr{U}(\mathscr{A})$ denotes the center of $\mathscr{U}(\mathscr{A})$. The present note is motivated by a desire to investigate the analogous situation for a von Neumann algebra factor acting on a seperable Hilbert space.

As a preliminary step we study the projections $\boldsymbol{P}(\mathscr{A})$ of a von Neumann algebra $\mathscr{A}$. Two projections E and F are said to be equivalent [4] (denoted by $E \sim F$) if and only if there exists an operator V in $\mathscr{A}$ such that $V^*V = E$ and $VV^* = F$. (Such an operator V is called a partial isometry, it maps the range of E isometrically onto the range of F.) It is shown that for a factor $\mathscr{A}$ there exists a norm continuous one parameter family $\mathrm{E}(\lambda)$, $0 \leqq \lambda \leqq 1$, of projections with initial point $E = \mathrm{E}(0)$ and terminal point $F = \mathrm{E}(1)$ if and only if $E \sim F$ and $I - E \sim I - F$, where I is the identity