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A Note on Product Measures and Representations of the Canonical Commutation Relations

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Abstract. There is a well-known theorem which states that a non-zero σ-finite left quasi-invariant measure on a σ-compact locally compact group G must be equivalent to left Haar measure. It is shown in this paper that there is a natural generalization of this fact to the case in which the group G is replaced by a product space, one factor of which is a group. With the aid of this generalization, an easy proof of the following fact, due to H. Araki, is given: the representations of the canonical commutation relations constructed in the usual measure-theoretic manner are ray continuous.

Almost invariably, the most desirable measure on a product space is a product measure. However, in doing integration theory on a product space, one is sometimes confronted with a measure which is *a priori* not even equivalent to a product measure. It is therefore of some interest to find conditions under which a measure on a product space must be equivalent to a product measure. Theorem 1 below gives such a condition which is, roughly speaking, that one of the factors be a group and that the measure in question be quasi-invariant under the action of the group on the product space. Araki proved Theorem 1 for the special case of a Euclidean group [1; Lemma 5.2]. His proof relies on the ray continuity of the representations of the canonical commutation relations (CCRs) constructed in the usual measure-theoretic manner (see [1; Section 1]). It was pointed out by Araki that the converse is also true, i.e., that ray continuity could be deduced from Theorem 1. In fact, the ray continuity is an easy consequence of Theorem 2, which itself depends on Theorem 1.

Suppose that G is a σ-compact locally compact group, that $\mathscr{A}$ is its σ-algebra of Borel sets (i.e., the σ-algebra generated by the open sets), and that λ is a left-invariant Haar measure on $(G, \mathscr{A})$. Suppose further that $\mathscr{B}$ is a σ-algebra of subsets of a non-empty set Z. Let $\mathscr{A} \times \mathscr{B}$ be the product σ-algebra on $G \times Z$. Measurability of subsets of or functions defined on G [resp., $Z, G \times Z$] will always be taken with respect to $\mathscr{A}$ [resp., $\mathscr{B}$, $\mathscr{A} \times \mathscr{B}$]. Setting $x(y, \zeta) = (xy, \zeta)$ for all $x, y \in G$ and all $\zeta \in Z$ defines a left action of G on $G \times Z$. The characteristic function of a subset S of $G \times Z$ will be denoted by 1_S.

Theorem 1. *Suppose that* ν *and* μ *are* σ*-finite measures on* $(G \times Z, \mathscr{A} \times \mathscr{B})$ *and* $(Z, \mathscr{B})$, *resp. Then* ν *is equivalent to* $\lambda \times \mu$ *if and only if*