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A Note on Correlations between Eigenvalues of a Random Matrix

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Abstract. Dyson's method is adopted here for the so called Gaussian ensembles. Incidently this confirms the long cherished belief that the statistical properties of a small number of eigenvalues is the same for the two kinds of ensembles, the circular and the Gaussian ones.

In studies of the statistical behaviour of the eigenvalues of random matrices, some authors have used as a basis the Gaussian matrix ensembles [1] while others have used the circular ensembles [2]. The Gaussian ensembles have a clearer physical motivation, while the circular ensembles are mathematically simpler. The choice of ensemble has been a matter of personal taste, and it has never been made clear how far the predictions of the theory might depend on the ensemble which is chosen. We here demonstrate that the predictions of Gaussian and circular ensembles in fact become identical in the limit as the order N of the matrices tends to infinity. More precisely, we prove that for any fixed n the joint probability density function of n eigenvalues in the Gaussian ensemble of order N tends to the same limit when $N \to \infty$ as the corresponding density function in the circular ensemble. This means that in an infinitely long eigenvalue sequence all the statistical properties are independent of the choice of ensemble.

Recently Dyson derived explicit analytical expressions for the joint probability density functions of n eigenvalues belonging to a random matrix taken from the circular ensembles [3]. We indicate below the necessary changes in his equations to give explicit expressions for the same correlation functions the random matrix being taken this time from the corresponding Gaussian ensembles. These changes do not alter in any way his arguments and we write therefore only the equations which are changed and which replace those in Dyson's paper with the same numbering. This note is supposed to be read along with Dyson's original.

$$x_j = \frac{\pi}{(2N)^{\frac{1}{2}}} \frac{1}{D} E_j, \tag{1.1}$$

$$Q_{N\beta}(x_1, \ldots, x_N) = C_{N\beta} \exp\left(-\frac{1}{2} \beta \sum_1^N x_j^2 \right) \prod_{j<k} |x_j - x_k|^\beta. \tag{1.2}$$