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Non-factor Quasi-free States of the CAR-algebra

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Abstract. A necessary and sufficient condition is given in order that a quasi-free state on the Clifford algebra $\overline{\mathscr{A}(H, s)}$ build on a real separable Hilbert space (H, s) be a factor state.

I. Introduction

Let (H, s) be a real Hilbert space which is separable (i.e. H is a real vector space and s a real scalar product on H). Let $\overline{\mathscr{A}(H, s)}$ be the CAR-algebra constructed on (H, s) i.e. it is the C^*-algebra generated by the elements $B(\psi)$ where $\psi \to B(\psi)$ is a real linear map of H into $\overline{\mathscr{A}(H, s)}$ satisfying the anticommutation relations

$$[B(\psi), B(\varphi)]_+ = 2s(\psi, \varphi)\, I$$

for all ψ and φ of H; I is the unit element in $\overline{\mathscr{A}(H, s)}$.

The quasi-free states ω_A on $\overline{\mathscr{A}(H, s)}$ are those states which are completely determined by an operator A on H such that for all $\psi, \varphi \in H$

$$\omega_A(B(\psi)\, B(\varphi)) = s(\psi, \varphi) + i\, s(A\psi, \varphi)\,, \tag{1}$$

$$s(A\psi, \varphi) = -s(\psi, A\varphi) \quad \text{or} \quad A^+ = -A\,, \tag{2}$$

$$\|A\| \leqq 1\,. \tag{3}$$

For more details see (1).

A state on a C^*-algebra is called factor state if it induces a factor G.N.S. representation. In this note we prove that ω_A is not a factor state if and only if the dimension of the kernel of A is odd and

$$\operatorname{Tr}[1 - (A^* A)^{\frac{1}{2}}] < \infty$$

II. The Theorem

Among the set of quasi-free states ω_A we distinguish two cases: let $\mathfrak{M}_A$ be the kernel of the operator A, then:

1. dimension of $\mathfrak{M}_A$ is even or infinite,
2. dimension of $\mathfrak{M}_A$ is odd.

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