Commun. math. Phys. 18, 265—274 (1970)

On Quadratic First Integrals of the Geodesic Equations for Type {22} Spacetimes

Martin Walker and Roger Penrose
Department of Mathematics, Birkbeck College, London, England

Received May 1, 1970

Abstract. It is shown that every type {*22*} vacuum solution of Einstein's equations admits a quadratic first integral of the null geodesic equations (conformal Killing tensor of valence 2), which is independent of the metric and of any Killing vectors arising from symmetries. In particular, the charged Kerr solution (with or without cosmological constant) is shown to admit a Killing tensor of valence 2. The Killing tensor, together with the metric and the two Killing vectors, provides a method of explicitly integrating the geodesics of the (charged) Kerr solution, thus shedding some light on a result due to Carter.

In a remarkable paper, Carter [1] has shown how to integrate the geodesic equations in a class of solutions of Einstein's equations including the (charged) Kerr solution [2], thus reducing the geodesic problem to one of quadratures. Carter's method depends on a peculiar feature of these solutions, namely that "the Hamilton-Jacobi equation (for the geodesic problem) can be solved by separation of variables", in a particular coordinate system [3]. In the present work we obtain an alternative procedure which achieves the same ends, but which is in some respects more transparent than Carter's. In addition, our method provides an explicit integration of all *null* geodesics in *any* type {*22*} vacuum solution. As yet, however, we have not been able to adapt our method so as to obtain charged particle orbits.

By a *(vacuum)* solution we understand a pair (M, g_{ab}) consisting of a 4-dimensional connected Hausdorff differentiable manifold M, and a pseudo-Riemannian metric g_{ab} of signature $(+ - - -)$ defined on M satisfying Einstein's (vacuum) gravitational field equations (with or without cosmological constant). In order to interpret the physical significance of a vacuum) solution, it is frequently useful to understand the global nature[1] of the spacetime (M, g_{ab}). The global analysis of a (vacuum) solution, however, is usually a difficult task since in most cases g_{ab} is given only locally. A knowledge of four first integrals of the geodesic

[1] For example, the nature of singularities [4] and conformal infinity [5].