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The Dirac Matrix Group and Fierz Transformations

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Abstract. The representation theory of the group generated by the Dirac matrices is studied. It is shown that the Fierz transformation can be expressed in terms of Racah coefficients of this group. A number of generalized Fierz transformations have been found. Simple rules are given for calculating Fierz invariants and anti-invariants.

1. Introduction

When performing explicit calculations on four-fermion interactions it is often expedient to apply the so-called Fierz transformation [1–3] and to make use of the Fierz invariants and anti-invariants [4–7]. The usual derivation of the explicit form of this transformation does not provide much insight in the underlying principles and is somewhat clumsy. Case [8] studied the Fierz identities in relation with the theory of spinor representations of orthogonal groups. He found for every orthogonal group the corresponding Fierz transformation using methods which do not differ essentially from the conventional procedure.

In a certain sense the Fierz transformation resembles crossing relations. Now it is well-known that the crossing matrix for isospin can be expressed as a Racah coefficient of the relevant group $SU(2)$ [9–12]. In this paper we study the group generated by the Dirac matrices and apply the well-known techniques of $SU(2)$, embodying elements such as Clebsch-Gordan coefficients (or Wigner coefficients), Racah coefficients, irreducible tensor operators and Wigner-Eckart theorem. This is possible, because this group is simply reducible. A crucial point will be the interpretation of the Γ-matrices as tensor operators. We show that the Fierz matrix is related to the Racah coefficients of the Dirac matrix group. Furthermore several other generalized Fierz transformations are given most of which do not seem to have been known before. The method with which we obtain these results gives more insight in the nature of the Fierz transformations and provides us with simple rules to calculate all kinds of Fierz matrices and the corresponding invariants and anti-invariants.