

The Even CAR-Algebra

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Abstract. It is shown that the even CAR-algebra over a separable Hilbert space is *-isomorphic to the CAR-algebra.

Let K be a separable infinite dimensional complex Hilbert space. Let $\mathfrak{A}(K)$ be the CAR-algebra over K . Then $\mathfrak{A}(K)$ is the C^* -algebra generated by elements $a(f)$, where $f \rightarrow a(f)$ is a linear map of K into $\mathfrak{A}(K)$ satisfying the canonical anticommutation relations

$$\begin{aligned} a(f)a(g)^* + a(g)^*a(f) &= (g, f)I, \\ a(f)a(g) + a(g)a(f) &= 0, \end{aligned}$$

for all $f, g \in K$, I denoting the identity operator in $\mathfrak{A}(K)$. Let γ be the *-automorphism of $\mathfrak{A}(K)$ such that $\gamma(a(f)) = -a(f)$ for all $f \in K$, and let $\mathfrak{A}(K)_e$ be the C^* -algebra of even elements in $\mathfrak{A}(K)$, i.e. $x \in \mathfrak{A}(K)$ if and only if $\gamma(x) = x$. It has been shown by Doplicher and Powers [1] that $\mathfrak{A}(K)_e$ is a simple C^* -algebra. In the present note we sharpen this result by showing that $\mathfrak{A}(K)_e$ is *-isomorphic to $\mathfrak{A}(K)$. We refer the reader to the thesis of Powers [3] for an account of the general theory of the CAR-algebra.

Theorem. $\mathfrak{A}(K)_e$ is *-isomorphic to $\mathfrak{A}(K)$.

Proof. Let f_1, f_2, \dots , be an orthonormal basis for K . Let K_n be the linear span of f_1, \dots, f_n , and $\mathfrak{A}(K_n)$ the CAR-algebra over K_n considered as a subalgebra of $\mathfrak{A}(K)$. Let $\mathfrak{A}(K_n)_e$ be the even subalgebra of $\mathfrak{A}(K_n)$. Since $\gamma(\mathfrak{A}(K_n)) = \mathfrak{A}(K_n)$ we clearly have $\mathfrak{A}(K_n)_e = \mathfrak{A}(K_n) \cap \mathfrak{A}(K)_e$. Let $U_i = I - 2a(f_i)^*a(f_i)$, $V_n = U_1 U_2 \dots U_n$. Then for $x \in \mathfrak{A}(K_n)$, $\gamma(x) = V_n x V_n$. Indeed, it suffices to show this for each $a(f_j)$, $j = 1, \dots, n$. But

$$V_n a(f_j) V_n = \prod_{i=1}^n U_i a(f_j) \prod_{i=1}^n U_i = U_j a(f_j) U_j = -a(f_j) = \gamma(a(f_j)).$$

Let P_n and Q_n be the spectral projections of V_n in $\mathfrak{A}(K_n)$, so that $V_n = P_n - Q_n$. Then P_n and Q_n are both projections of dimension 2^{n-1} in the $2^n \times 2^n$