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Stable Potentials I

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Abstract. We discuss a conjecture of Ruelle concerning *stable* potentials on a group. For the groups $\boldsymbol{Z}_2, \boldsymbol{Z}_3, \boldsymbol{Z}_4$, and $\boldsymbol{Z}_6$ any stable potential can be written as the sum of a non-negative function and a function of non-negative type. This is not true for the groups $\boldsymbol{Z}_k$ (k odd, $\geqq 5$). For the Euclidean group $\boldsymbol{R}^\nu$ the question is open.

§ 1

The following theorem is due to David Ruelle [1]. Let φ be a real valued, even, upper semicontinuous function on a Euclidean space $\boldsymbol{E}$. Let

$$\begin{cases} U_1 = 0 \\ U_n = \sum_{1 \leqq i < j \leqq n} \varphi(x_i - x_j) \end{cases} \qquad (n = 2, 3, \ldots) \tag{1}$$

The following conditions are equivalent:

$$\text{(a)} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(x_i - x_j) \geqq 0 \tag{2}$$

for all $n \geqq 1$ and all $(x_1, \ldots, x_n)$ in $\boldsymbol{E}^n$.

(b) There is a constant B such that

$$U_n(x_1, \ldots, x_n) \geqq -nB \tag{3}$$

for all $n \geqq 1$ and all $(x_1, \ldots, x_n)$ in $\boldsymbol{E}^n$.

(c) For all bounded Lebesgue measurable sets $\Lambda \subset \boldsymbol{E}$ and all positive numbers z and β the series

$$\Xi = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n e^{-\beta U_n} \tag{4}$$

converges.

The importance of this theorem is that the quantity Ξ has a fundamental significance in the statistical mechanics of classical systems in thermal equilibrium (it is the Grand Partition Function of Gibbs). φ is called the two-particle potential function, or simply the potential,

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