Commun. math. Phys. 13, 68—72 (1969)

Uniqueness of the Hamiltonian in Quantum Field Theories

STEPHEN PARROTT

Department of Physics, University of Mass.-Boston

Received June 25, 1968

Abstract. In most quantum field theories, one defines the Hamiltonian (energy) operator H as a limit of "cutoff" operators H_s: $H = \lim_{s \to \infty} H_s$. (The operator H_s would be the correct Hamiltonian for a world in which all momenta are smaller than s.) Since the cutoff operators seldom converge in any of the standard operator topologies, it is often necessary to invent more subtle notions of "convergence". For some of the these, it is not obvious that the "limit" operator H is unique. In this note we point out that for one such method of obtaining convergence, the "limit" operator is *not* unique. In fact, (under mild assumptions about the operators H_s), if H_s converges to H, then H_s also converges to $H + R$, where R is an arbitrary bounded positive operator.

0. Notation

Let $\mathscr{K}$ be a separable Hilbert space with inner product $(\cdot\,,\cdot)$. An *operator* H on $\mathscr{K}$ is a densely-defined linear transformation from $\mathscr{K}$ into $\mathscr{K}$ with domain $\mathscr{D}(H)$. We write $H' \subset H$ to mean $\mathscr{D}(H') \subset \mathscr{D}(H)$ and $H'f = Hf$ for all f in $\mathscr{D}(H')$. A *symmetric* operator satisfies: $(Hf, g) = (f, Hg)$ for all f, g in $\mathscr{D}(H)$. A symmetric operator is *essentially self-adjoint* if it has a unique self-adjoint extension. We assume that the reader is familiar with the basic facts concerning unbounded self-adjoint operators [cf. 4, Chap. 8].

1. Statement of the Problem

Suppose we are given a family of self-adjoint operators H_s, $(0 < s < \infty)$. Here are two related methods for obtaining a symmetric operator H as a "limit" of the family of operators H_s:

Method A. Find a dense linear manifold $\mathscr{D}$ and bounded invertible operators T_s, T such that:

(i) For all s, $T_s \mathscr{D} \subset \mathscr{D}(H_s)$.

(ii) For each f in $\mathscr{K}$, $\lim_{s \to \infty} T_s f = Tf$.

(iii) For all f in $\mathscr{D}$, $\lim_{s \to \infty} H_s T_s f$ exists.

Define the limit operator H with $\mathscr{D}(H) = T\mathscr{D}$ by:

$$HTf = \lim_{s \to \infty} H_s T_s f, \; f \quad \text{in} \quad \mathscr{D}\,.$$

The operator H is symmetric, but need not be self-adjoint.