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$\mathrm{PSL}_2(q)$ AND EXTENSIONS OF $\mathbf{Q}(x)$

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INTRODUCTION

We recall that an extension $K/\mathbf{Q}(x)$ of finite degree n is called *regular* if $\mathbf{Q}$ is algebraically closed in K. Tensoring with the complex field $\mathbf{C}$ we obtain the extension $\overline{K}/\mathbf{C}(x)$ of degree n. Since $\mathbf{C}(x)$ is the function field of the Riemann sphere $\mathbf{P}^1$, the field extension $\overline{K}/\mathbf{C}(x)$ corresponds to a cover $X \to \mathbf{P}^1$ of Riemann surfaces. All but finitely many points of $\mathbf{P}^1$ have exactly n preimages in X, and these finitely many exceptional points are called the *branch points*.

If a finite group G is the Galois group of a regular extension of $\mathbf{Q}(x)$, then G occurs as a Galois group over every number field (by Hilbert's irreducibility theorem). This is the basis of all recent work on the inverse problem of Galois theory (by Fried, Matzat, Thompson, and others). An important invariant of a regular extension of $\mathbf{Q}(x)$ is the number r of branch points. Most work has been concentrated on the case $r = 3$, using Thompson's concept of rigidity [Th1]. Indeed, the case $r = 3$ seems at first the natural one to work with, since it involves using $r - 1 = 2$ generators of the group under consideration, and it is known that every (finite) simple group can be generated by two elements; furthermore, the rigidity condition becomes too stringent for $r > 3$, and there seems only one example known (due to Thompson) of a simple group for which rigidity holds with $r > 3$.

It appears that for $r > 3$ one has to include the action of the Hurwitz monodromy group (see §2), which goes back to [Fr1]. Indeed Matzat [Ma1, Chapter III] has used this action to realize a few groups as Galois groups over $\mathbf{Q}(x)$ with $r > 3$. However, by far the most simple groups (and related groups) that are known so far to be Galois groups over $\mathbf{Q}$ (or over certain number fields) have

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