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The averaged moduli of smoothness with applications in numerical methods and approximation, by Blagovest Sendov and Vasil A. Popov. John Wiley & Sons, New York, 1988, 180 pp. ISBN 0-471-91952-7

For functions f, bounded (and Lebesgue measurable) on a compact interval $[a, b]$ of the real axis, the classical modulus of continuity (smoothness) may be introduced via

$$(1) \qquad \omega_k(f, \delta)_\infty := \sup_{a \le x \le b} \omega_k(f, x, \delta),$$

using the kth local modulus of continuity

$$\omega_k(f, x, \delta) := \sup\left\{|\Delta_h^k f(t)| : t, t + kh \in [a, b] \cap \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\},$$

$$\Delta_h^k f(t) := \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(t + jh).$$

It is well known that (1) as well as its L^p-analogue $(1 \le p < \infty)$

$$(2) \qquad \omega_k(f, \delta)_p := \sup_{0 \le h \le \delta} \left[\int_a^{b-kh} |\Delta_h^k f(x)|^p \, dx\right]^{1/p}$$

serve as a measure of smoothness of functions in many fields of analysis. In particular, the moduli (1; 2) often supply appropriate bounds for the error of approximation processes, given via sequences of bounded (e.g., integral) operators on L^p. When dealing with approximation procedures of a discrete structure, however, an estimation of L^p-errors (e.g., for Bernstein polynomials of bounded functions) versus $\omega_k(f, \delta)_p$ is not possible since, roughly speaking, $\omega_k(f, \delta)_p$ represents a bounded (sublinear) functional on L^p, whereas point evaluation functionals cannot be bounded with regard to the L^p-metric, even when restricted to continuous functions.

In this connection the authors offer the averaged or τ-modulus of smoothness

$$(3) \qquad \tau_k(f, \delta)_p := \left[\int_a^b (\omega_k(f, x, \delta))^p \, dx\right]^{1/p}$$