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Elementary geometry in hyperbolic space, by Werner Fenchel. De Gruyter Studies in Mathematics, vol. 11, Walter de Gruyter, Berlin, New York, 1989, xi+225 pp., $69.95. ISBN 0-89925-493-4

To obtain a helpful overview of the material in hand it is appropriate to begin with a brief discussion of the Möbius group in n-dimensions. Detailed accounts have been given from different perspectives by Ahlfors [1], Beardon [2], and Wilker [5].

Let $\Sigma = \Sigma^n = \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$ be the unit n-sphere in $\mathbf{R}^{n+1}$, $n \geq 2$. An $(n-1)$-sphere $\gamma = \gamma^{n-1}$ on Σ is the section of Σ by an n-flat containing more than one point of Σ and each such $(n-1)$-sphere determines an involution $\gamma : \Sigma \to \Sigma$ called inversion in γ. This inversion fixes the points of γ and interchanges other points in pairs which are separated by γ and have the property that any two circles of Σ, which pass through one of the points and are perpendicular to γ, meet again at the other point. The group generated by the set of all inversions of Σ is the n-dimensional Möbius group $\mathscr{M}_n$. Further properties of $\mathscr{M}_n$ can be inferred from the fact that it can also be defined as the group of bijections of Σ that preserves circles, or angles, or cross ratios, where a typical cross ratio of the four distinct points a, b, c, d belonging to Σ is the number $(\|a-b\|\,\|c-d\|)/(\|a-c\|\,\|b-d\|)$.

Let $\Pi = \Pi^n = \mathbf{R}^n \cup \{\infty\}$. Stereographic projection from Σ to Π transfers the Möbius group $\mathscr{M}_n$ to Π where it is natural to think of it as the group generated by reflections in $(n-1)$-flats and inversions in $(n-1)$-spheres. All the essential properties of the action of $\mathscr{M}_n$ are preserved in the transfer to Π because stereographic projection is induced by an inversion one dimension higher. Thus Σ and Π provide useful alternative models for viewing inversive n-space from a Euclidean perspective; to enter into the full spirit of their equivalence one need only remember that an inversive m-sphere in Π, $1 \leq m \leq n-1$, can equally well mean a Euclidean m-sphere or a Euclidean m-flat augmented by the point ∞. Since Σ^n and Π^n sit naturally in Π^{n+1}, we can perform extensions of their transformations and regard the copies of $\mathscr{M}_n$ associated with them as conjugate subgroups of $\mathscr{M}_{n+1}$. The extension of a Möbius