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EXOTIC COHOMOLOGY AND INDEX THEORY

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1. INTRODUCTION

The main result of this paper is a version of the Atiyah-Singer index theorem for Dirac-type operators on (noncompact) complete Riemannian manifolds. The statement of the theorem involves a novel "cohomology" theory for such manifolds. This theory, called *exotic cohomology*, depends on the structure at infinity of a space; more precisely, it depends on the way that large bounded sets fit together. For each cohomology class in this theory, we define a "higher index" of a Dirac-type operator, enjoying the stability and vanishing properties of the usual Atiyah-Singer index; these higher indices are analogous to the Novikov higher signatures. Our main theorem will compute these higher indices in terms of standard topological invariants. Applications of this index theorem include a different approach to some of the results of Gromov and Lawson [10] on topological obstructions to positive scalar curvature.

The concept of index that we will use involves the K-theory functors K_0 and K_1 for operator algebras [3, 14]. Suppose that B is an ideal in a unital algebra C, and let $T \in C$ be invertible modulo B. (In the classical Atiyah-Singer index theorem, one takes C to be the bounded operators on the L^2 space of some compact manifold, B the compact operators, and T an elliptic pseudodifferential operator of order zero.) Then T has an "index" in the K-theory group $K_0(B)$ (in the classical case this is just $\mathbf{Z}$, and one recovers the usual Fredholm index). Now let M be a complete Riemannian manifold, possibly noncompact. In [18] I introduced an algebra $\mathscr{U}(M)$ which is defined as follows: $\mathscr{U}(M)$ consists of all bounded operators A on $L^2(M)$ that have a kernel representation

$$Au(x) = \int k(x, y)u(y)\,dy,$$

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