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NONRECURSIVE FUNCTIONS IN REAL ALGEBRAIC GEOMETRY

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ABSTRACT. As a result of the Tarski-Seidenberg theorem, problems in real algebraic geometry usually have constructive solutions. In this article we show that this is not always the case. We consider the following problem, which is of interest for its own sake.

Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a nonsingular compact algebraic surface of degree d. Let Σ^n be isotopic to the standard hypersphere $S^n \subset \mathbf{R}^{n+1}$. It is well known that it is not always possible to connect Σ^n and S^n by an isotopy passing via nonsingular algebraic hypersurfaces of degree not higher than d. We prove that it is always possible to connect Σ^n and S^n by an isotopy passing via algebraic surfaces of some degree A which depends on n and d only. Consider for each n the smallest of such degrees as a function $A_n(d)$. What can be said about these functions?

We prove that for each $n \geq 5$, the function $A_n(d)$ cannot be majorized by a recursive function of d. Also, some generalizations of these results are stated below.

1. Algebraic approximation of isotopies. It is well known that a compact hypersurface in a Euclidean space may be approximated by a nonsingular algebraic hypersurface. It is possible to prove that an isotopy between two compact nonsingular algebraic hypersurfaces may be approximated by an isotopy which passes via nonsingular algebraic hypersurfaces only. This can be generalized for regular complete intersections of compact nonsingular algebraic hypersurfaces. More precisely, let $p_i(x)$, $i \in \{1, \ldots, k\}$ be a system of polynomials on $\mathbf{R}^{n+k}$ such that for all i the set Z_i of zeroes of $p_i(x)$ is compact and not empty and the Jacobi matrix $J(p_1(x), p_2(x), \ldots, p_k(x))$ has maximal rank at each point of $Z_p = Z_1 \cap Z_2 \cap \cdots \cap Z_k$. If Z_p is nonempty, then we call $\{p_i(x)\}$ *a regular system* and Z_p *a compact complete intersection*, or briefly a CCI (cf. **[1, 4]** and references there for properties of compact complete intersections). If some CCI $M^n \subset \mathbf{R}^{n+k}$ is representable as a regular intersection of nonsingular compact algebraic hypersurfaces of degrees less than or equal to d, then we shall call M^n a d-CCI. Let $\Omega_{n,k}(M^n)$ denote the space of all n-dimensional submanifolds of $\mathbf{R}^{n+k}$, diffeomorphic to M^n, with the C^1-topology. We shall denote by $T_n(\mathbf{R}^{n+k})$ the space $\bigvee \Omega_{n,k}(M^n)$, where the disjoint union is taken over all nondiffeomorphic compact C^∞-smooth manifolds M^n embeddable in $\mathbf{R}^{n+k}$, and by $T_n^{(d)}(\mathbf{R}^{n+k})$ its subspace consisting of all d-CCI's. We shall say that two

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