has become the first part of a two volume treatise. The second volume is intended to cover the work on the quantised systems up to the current time.

ROGER DODD
HIROSHIMA UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Moduli of smoothness, by Z. Ditzian and V. Totik. Springer Series in Computational Mathematics, Springer-Verlag, New York, Berlin and Heidelberg, 1987, ix+225 pp., $54.90. ISBN 0-387-96536-x

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, the domain of which is a (bounded) interval D, they essentially measure the structure or smoothness of f via the rth (symmetric) difference

$$\Delta_h^r f(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + rh/2 - kh)$$

(with the convention that $\Delta_h^r f(x) = 0$ if $x \pm rh/2 \notin D$). In fact, for functions f belonging to the Lebesgue space $L^p(D)$, $1 \leq p < \infty$, or the space $C(D)$ $(p = \infty)$ of continuous functions, the classical rth modulus

$$(1) \qquad \omega^r(f,t)_p := \sup_{|h| \leq t} \|\Delta_h^r f\|_p$$

has turned out to be a rather good measure for determining the rate of convergence of best approximation or of particular linear approximation processes.

For example, for 2π-periodic functions f, D. Jackson (1911) and S. N. Bernstein (1912) showed that the error of best approximation $E_n^*(f)_p$ by trigonometric polynomials of degree at most n has the same rate of convergence as the rth modulus in the sense that, for $0 < \alpha < r$,

$$(2) \qquad \omega^r(f,t)_p = \mathscr{O}(t^\alpha) \qquad (t \to 0) \Leftrightarrow E_n^*(f)_p = \mathscr{O}(n^{-\alpha}) \qquad (n \to \infty).$$

In the case of algebraic approximation, however, that is by algebraic polynomials $p_n \in \mathscr{P}_n$ of degree at most n, this result is no longer true. Though one has here the direct estimate

$$E_n(f)_p := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_p \leq K\omega^r \left(f, \frac{1}{n}\right)_p,$$

given in the doctoral thesis of D. Jackson (1911), it was observed by S. M. Nikolskii (1946) that for functions f satisfying

$$(3) \qquad \omega^r(f,t)_p = \mathscr{O}(t^\alpha) \qquad (0 < \alpha < r),$$

the polynomial $p_n^* \in \mathscr{P}_n$ of best approximation has a faster rate of convergence near the boundary of D than in the interior. In fact, it was the Russian school in approximation, in particular A. F. Timan (1951) and V. K. Dzjadyk (1959), that succeeded in characterizing (3) in terms of algebraic polynomials for