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THE QUANTUM MECHANICAL SPHERICAL PENDULUM

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ABSTRACT. In this announcement we describe the asymptotic behavior of the spectrum of the quantum mechanical spherical pendulum as Planck's constant tends to zero.

We begin by discussing

1. The classical spherical pendulum [6, 4]. As a Hamiltonian system the spherical pendulum has a configuration space

$$S^2 = \{q = (q_1, q_2, q_3) \in \mathbf{R}^3 | 1 = q_1^2 + q_2^2 + q_3^2 = \langle q, q\rangle\}$$

and a phase space

$$T^*S^2 = \{(q,p) \in \mathbf{R}^3 \times \mathbf{R}^3 | \langle q,q\rangle = 1, \langle q,p\rangle = 0\}.$$

The standard symplectic form $\sum_{i=1}^3 dq_i \wedge dp_i$ on $\mathbf{R}^3 \times \mathbf{R}^3$ when restricted to T^*S^2 gives the canonical symplectic form on T^*S^2. The dynamics is given by the Hamiltonian function

$$E: T^*S^2 \to \mathbf{R}: (q,p) \to \tfrac{1}{2}\langle p,p\rangle + q_3.$$

Since E is invariant under rotations about the q_3 axis lifted to T^*S^2, the function

$$L: T^*S^2 \to \mathbf{R}: (q,p) \to q_1p_2 - q_2p_1$$

is an integral of the Hamiltonian vectorfield X_E. L is the q_3-component of angular momentum. Thus the flows ϕ_t^E and ϕ_t^L of X_E and X_L, respectively, commute. Hence the spherical pendulum is completely integrable.

Consider the energy momentum mapping

$$\mathscr{EM}: T^*S^2 \to \mathbf{R}^2: (q,p) \to (E(q,p), L(q,p))$$

(see Figure 1). Suppose that $(e,l) \in \mathscr{R}$, the set of regular values of $\mathscr{EM}$, which is the shaded region in Figure 1 excluding $(1,0)$ and boundary curves. Then $\mathscr{EM}^{-1}(e,l) = E^{-1}(e) \cap L^{-1}(l)$ is a compact, connected, smooth, two-dimensional submanifold of T^*S^2. On $\mathscr{EM}^{-1}(e,l)$ we have an $\mathbf{R}^2$ action defined by

$$\Phi: \mathbf{R}^2 \times \mathscr{EM}^{-1}(e,l) \to \mathscr{EM}^{-1}(e,l): ((t_1,t_2),m) \to \phi_{t_1}^E \circ \phi_{t_2}^L(m)$$

which is transitive. Therefore the isotropy group

$$P(e,l) = \left\{(T_1,T_2) \in \mathbf{R}^2 | \phi_{T_1}^E \circ \phi_{T_2}^L = \mathrm{id}_{\mathscr{EM}^{-1}(e,l)}\right\}$$

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