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It is well known that the parametrix of an elliptic differential operator on a manifold without boundary will be a pseudodifferential operator. When one considers an elliptic boundary value problem the parametrix certainly contains a pseudodifferential operator in the domain, but other terms arise because of the presence of the boundary. In the case of manifolds without boundary it is useful to consider an algebra of pseudodifferential operators that contain both the elliptic differential operators and their parametrices. The study of the index of an elliptic operator is one of the applications of such algebra.

Analogously one may try to construct an algebra of operators on a manifold with boundary that contains both elliptic boundary value problems and their parametrices. The first question to answer is the following: how to define a Fredholm boundary value problem for a pseudodifferential operator on a manifold with a boundary?

Consider first an elliptic boundary value problem in the domain $\Omega \subset \mathbf{R}^n$:

$$L(x, D)u = f, \qquad x \in \Omega, \tag{1}$$
$$p_{\partial\Omega} B(x, D)u = g, \tag{2}$$

where $p_{\partial\Omega}$ is the restriction operator to $\partial\Omega$, and $L(x, D), B(x, D)$ are, in general, matrices of differential operators. "Freeze" the coefficients of (1), (2) at an arbitrary point $x' \in \partial\Omega$, introduce a system of coordinates in a neighbourhood of x' such that the equation of $\partial\Omega$ will be $x_n = 0$, and, ignoring the dependency of coefficients on x', take the Fourier transform in variables tangential to $\partial\Omega$. Then one can associate with (1), (2) a family of boundary value problems for ordinary differential equations on the half-line

$$L_0(y, 0, D_n)v(x_n) = f_0(x_n), \qquad 0 < x_n < +\infty, \tag{1'}$$
$$B_0(y, x_n, D_n)v(x_n)|_{x_n=0} = g_0, \tag{2'}$$

where $y = (x', \xi') \in T^*(\partial\Omega)$, $|\xi'| = 1$, and $L_0(y, x_n, \xi_n)$, $B_0(y, x_n, \xi_n)$ are symbols of the principal parts of L, B written in coordinates (x', x_n). $T^*(\partial\Omega)$ is the cotangent bundle of $\partial\Omega$. The investigation of (1′), (2′) is the crucial step in the study of boundary value problem (1), (2). Note that (1′) defines a Fredholm operator from $H_s(\mathbf{R}^1_+)$ to $H_{s-2m}(\mathbf{R}^1_+)$, where $2m$ is the order of L_0 and $H_s(\mathbf{R}^n_+)$ is the Sobolev space in $\mathbf{R}^1_+$. This Fredholm operator has no cokernel and the dimension of its kernel is exactly the number of boundary conditions (2′). Indeed the role of the boundary conditions (2) is to "kill" the kernel of (1′).

Consider now a pseudodifferential equation in $\Omega \subset \mathbf{R}^n$:

$$p_\Omega A(x, D)u_0 = f, \tag{3}$$