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ON THE LOCAL SEVERI PROBLEM

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Introduction. We study plane curves with singularities. Let $\mathbf{P}^N$ be the projective space parametrizing plane curves of degree n ($N = n(n+3)/2$). Let $V(n,g) \subset \mathbf{P}^N$ be the locus of reduced irreducible plane curves of degree n and (geometric) genus g, and $l \subset \mathbf{P}^2$ a fixed line. Following Zariski [**7**], we consider the subvariety $Z(n,g) \subset \overline{V(n,g)}$ of curves which contain l as a component. The purpose of this note is to study $Z(n,g)$ and prove the following

THEOREM. *Let $\mathscr{E}(n,g)$ be a branch of $\overline{V(n,g)}$ through a point of $Z(n,g)$ corresponding to a reduced curve. Then the general members of $\mathscr{E}(n,g) \cap Z(n,g)$ have only nodes as singularities.*

It is well known (cf. Severi [**5**, §11]) that this Theorem implies the following fundamental result of Harris.

COROLLARY (HARRIS [**3**]). *$V(n,g)$ is irreducible.*

In the case when $L \in \mathscr{E}(n,g) \cap Z(n,g)$ is a union of n distinct lines passing through a point, our theorem is a realization of Severi's attempt to prove that L can be regenerated to a reducible nodal curve of $\mathscr{E}(n,g)$ [**5**, §11, p. 344]. The idea of using decreasing induction on g and equations of curves in the proof was suggested in Zariski [**7**]. On the other hand, Harris [**3**] and Ran [**4**] use the degeneration method in their treatment of plane curves.

Proof of Theorem. We set $d = (n-1)(n-2)/2 - g$ and $\nu(n,d) = \dim V(n,g) = 3n + g - 1$ ([**5**, §11], [**6**]). Let $\Sigma_{n,d} \subset \mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2)$ be the closure of the locus of irreducible curves of degree n with d nodes and no other singularities, and π_N the projection to $\mathbf{P}^N$. Given a pair consisting of a reduced curve $E \in \overline{V(n,g)}$ and a branch of $\overline{V(n,g)}$ through the curve, one can define, via π_N, an element of $\mathrm{Sym}^d(\mathbf{P}^2)$, called the cycle of assigned singularities of the pair. Our basic tool is the dimension-theoretic characterization of maximal families of nodal curves by Arbarello and Cornalba [**1**] and Zariski [**6**] and its generalization by Harris [**3**, Proposition 2.1].

Let C be a general member of $\mathscr{E}(n,g) \cap Z(n,g)$. We will prove that C is nodal and all its unassigned nodes lie on l for every choice of a branch of $\mathscr{E}(n,g)$ through C.

LEMMA. *For $d \leq 3$, $\Sigma_{n,d}$ is irreducible and unibranch.*

PROOF OF THE LEMMA. Let Σ', $\Sigma'' \subset \Sigma_{n,d}$ be components such that a general member of Σ' has d nodes in general position. A dimension count

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