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BI-INVARIANT SCHWARTZ MULTIPLIERS AND LOCAL SOLVABILITY ON NILPOTENT LIE GROUPS

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Let X denote a finite-dimensional vector space with a fixed positive definite inner product, and let $\mathcal{S}(X)$ denote the Schwartz space on X. We let $\mathcal{MS}(X)$ denote the space of continuous endomorphisms of $\mathcal{S}(X)$ that commute with the action of X on $\mathcal{S}(X)$. The elements of $\mathcal{MS}(X)$ are given by convolution by tempered distributions; i.e., for $E \in \mathcal{MS}(X)$ there is a $D_E \in \mathcal{S}^*(X)$ such that $Ef(x) = \langle D_E, l_x \check{f}\rangle := D_E * f(x)$, where $\check{f}(x) = f(-x)$ and $l_x f(y) = f(y-x)$. Conversely, if $D \in \mathcal{S}^*(X)$, then one can easily see that $E_D\colon f \to D * f$ is a mapping of $\mathcal{S}(X)$ into the smooth functions on X that commutes with translation. Schwartz [S] shows that $E_D \in \mathcal{MS}(X)$ if and only if $\hat{D}$, the Fourier transform of D, is given by a smooth function on X^* which has polynomial bounds on all derivatives. In this note we announce analogues of these results for arbitrary nilpotent Lie groups. Complete proofs will appear elsewhere.

Let N denote a connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{n}$. The exponential mapping, $\exp\colon \mathfrak{n} \to N$, is a diffeomorphism, and in terms of the corresponding coordinates left and right translation on N are polynomial mappings. Thus, if $\mathcal{S}(N)$ denotes the image under composition with exp of $\mathcal{S}(\mathfrak{n})$, the right and left action of N on $\mathcal{S}(N)$ are continuous endomorphisms, where $\mathcal{S}(N)$ is topologized so that composition with exp is an isomorphism from $\mathcal{S}(\mathfrak{n})$ to $\mathcal{S}(N)$. We denote by $\mathcal{S}^*(N)$ the dual of $\mathcal{S}(N)$, the space of tempered distributions on N.

For $f \in \mathcal{S}(N)$, the Fourier transform of f, $\hat{f}$, is defined on $\mathfrak{n}^*$, the dual of $\mathfrak{n}$, by

$$\hat{f}(\xi) = \int_{\mathfrak{n}} f(\exp X) e^{-2\pi i\langle \xi, X\rangle}\, dX.$$

One has that $f \to \hat{f}$ is an isomorphism from $\mathcal{S}(N)$ onto $\mathcal{S}(\mathfrak{n}^*)$. For $D \in \mathcal{S}^*(N)$, $\widehat{D}$ is defined on $\mathcal{S}(\mathfrak{n}^*)$ by $\langle \widehat{D}, f\rangle = \langle D, \hat{f} \circ \log\rangle$, where log denotes the inverse of exp.

Let Ad^* denote the coadjoint representation of N on $\mathfrak{n}^*$. A tempered distribution D on $\mathfrak{n}^*$ is said to be Ad^*-invariant if $\langle D, f \circ \mathrm{Ad}^*\, x\rangle = \langle D, f\rangle$ for all $x \in N$ and $f \in \mathcal{S}(\mathfrak{n}^*)$. A tempered distribution D on N is said to be bi-invariant if $\langle D, \mathfrak{r}_{x^{-1}} f\rangle = \langle D, l_x f\rangle$ for all $f \in \mathcal{S}(N)$, where $\mathfrak{r}_x f(y) = f(yx)$ and $l_x f(y) = f(x^{-1}y)$ for all $x, y \in N$. A straightforward computation shows that an element $D \in \mathcal{S}(N)$ is bi-invariant if and only if $\widehat{D}$ is Ad^*-invariant.

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