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THE NUMBER OF EQUATIONS NEEDED TO DEFINE AN ALGEBRAIC SET

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Let B be a commutative Noetherian ring, $X = \operatorname{Spec} B$ the associated affine scheme, $I \subset B$ an ideal and $V = V(I) \subset X$ the closed subset defined by I.

DEFINITION. *Elements $f_1, \ldots, f_s \in I$ define V set-theoretically (equivalently, V is defined set-theoretically by s equations $f_1 = 0$, $f_2 = 0, \ldots, f_s = 0$) if $\sqrt{(f_1, \ldots, f_s)} = \sqrt{I}$.*

Hilbert's Nullstellensatz implies that in the case when B is a finitely generated algebra over an algebraically closed field k, this definition agrees with the usual one, i.e., all $f_1, \ldots, f_s$ vanish at a k-rational point if and only if it belongs to V. In the sequel "defined" always means "defined set-theoretically".

The question we are dealing with here concerns the minimum number of equations needed to define a given $V \subset X$. A classical result that goes back to L. Kronecker **[Kr]** says that if B is n-dimensional, then $n+1$ equations would suffice for every $V \subset X$. Our first theorem describes those $V \subset X$ which can be defined by n equations.

THEOREM A. *Let k be an algebraically closed field, X a smooth affine n-dimensional variety over k with coordinate ring B, and $V = V' \cup P_1 \cup P_2 \cup \cdots \cup P_r$ an algebraic subset of $X = \operatorname{Spec} B$, where V' is the union of irreducible components of positive dimensions and $P_1, P_2, \ldots, P_r$ some isolated closed points (which do not belong to V'). Then V can be defined by n equations if and only if one of the following conditions holds.*

(i) *$r = 0$, i.e., V consists only of irreducible components of positive dimension.*

(ii) *V' is empty, i.e., V consists only of closed points and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1P_1 + n_2P_2 + \cdots + n_rP_r = 0$ in $A_0(X)$.*

(iii) *V' is nonempty, $r \geq 1$ and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1P_1 + n_2P_2 + \cdots + n_rP_r$ belongs to the image of the natural map $A_0(V') \to A_0(X)$ induced by the inclusion $V' \to X$.*

Here $A_0(\cdot)$ stands for the group of zero-cycles modulo rational equivalence **[Fu]**.

SKETCH OF PROOF. Our proof consists of three steps. In Step 1 we construct an ideal $I \subset B$ such that $\sqrt{I}$ is the defining ideal of V, and in addition I has some other special properties. In Step 2 we, in a special way, pick some ideals $Q_1, \ldots, Q_n$ such that $\sqrt{Q_i}$ is a maximal ideal containing I for each i and J/J^2 is n-generated, where $J = I \cap Q_1 \cap \cdots \cap Q_n$. In Step 3 we

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