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NUMERICALLY DETERMINING SOLUTIONS OF SYSTEMS OF POLYNOMIAL EQUATIONS

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In this report we suggest some efficient algorithms for numerically determining all solutions of a system of n polynomial equations in n unknowns. Such systems are common in many fields of engineering. When all equations are linear, there is at most one isolated solution. In the general case, even the number of solutions can be difficult to predict.

By a classical theorem of Bézout, the number of isolated solutions of the system is bounded above by the *total degree* $d = d_1 \cdots d_n$, where d_i denotes the degree of the ith equation. Empirically, we find that most systems arising in applications have fewer than d solutions. We call such systems *deficient*. Our purpose is to describe some methods for which the computational work, instead of being proportional to the total degree, is proportional to the actual number of solutions.

The first practical computer-implementable method for numerically solving polynomial systems was introduced in [**D**] (see also [**GZ**]) using arguments based on algebraic geometry. The authors of [**CMY**] reformulated this result, replacing the algebraic geometry arguments with a general version of Sard's Theorem. The article [**AG**] presents a survey of homotopy methods for numerically solving systems of equations.

Elimination theory is the classical approach to solving systems of polynomial equations, but its reliance on symbolic manipulation makes it seem unsuitable for all but small problems. Moreover, the method (unlike Gaussian elimination for the linear case) reduces the problem to the ill-conditioned problem of numerically solving a high-degree polynomial equation in one variable. In this paper, we use elimination theory and other techniques from algebraic geometry as theoretical tools, but our algorithms avoid the computing of resultants.

1. Random product homotopy. Let

$$\begin{gathered} p_1(x_1, \dots, x_n) = 0, \\ \vdots \\ p_n(x_1, \dots, x_n) = 0 \end{gathered}$$

be the system of polynomial equations to be solved, i.e., $P\colon \mathbf{C}^n \to \mathbf{C}^n$, $P = (p_1, \dots, p_n)$. Although we are only interested in computing the solutions in

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