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CONTROL VARIATIONS WITH AN INCREASING NUMBER OF SWITCHINGS

MATTHIAS KAWSKI

1. Introduction. The purpose of this paper is to introduce new families of control variations and exhibit how they lead to high-order conditions for controllability which cannot be obtained by the usual methods. Also, we explain why the underlying phenomenon is likely to be very important for the synthesis of (time-optimal) feedback.

Suppose $X(x)$ and $Y(x)$ are real analytic vectorfields on R^n with $X(0) = 0$. They give rise to the single-input affine control system

$$(1) \qquad \begin{cases} \dot{x} = X(x) + uY(x), & |u(t)| \leq \varepsilon_0, \\ x(0) = 0, \end{cases}$$

where the control u is a measurable function defined on some interval $[0, T]$ with bound $\varepsilon_0 > 0$. The solution to (1) with control u is denoted by $x(t, u)$. The *attainable set* at time t (with control bound ε_0) is $\mathcal{A}_{\varepsilon_0}(t) = \{x(t,u) : |u(\cdot)| \leq \varepsilon_0\}$. The system (1) is *small-time locally controllable* (STLC) if $\mathcal{A}_{\varepsilon_0}(t)$ contains the rest solution $x \equiv 0$ in its interior for all ε_0, $t > 0$.

Let $L(Y, X)$ be the Lie algebra generated by the vectorfields Y and X, and $L(Y,X)(p) = \{W(p) : W \in L(Y,X)\}$ for a point $p \in R^n$. A consequence of the *Hermann-Nagano Theorem* is [**13**]: If $L(Y,X)(0)$ is the full tangent space at zero then $\operatorname{int} \mathcal{A}_{\varepsilon_0}(t) \neq \emptyset$ for all ε_0, $t > 0$, and in the case of analytic vectorfields the converse is true, also. Sometimes referred to as the *Second Nagano Theorem* is [**10**], loosely speaking: Up to diffeomorphisms all local properties of (1) are determined by the values of the iterated Lie brackets of X and Y at zero. In view of this it is natural to look for necessary and sufficient conditions for STLC in terms of Lie brackets of Y and X at 0. In recent years substantial progress in this direction has been made, e.g. [**2, 4, 5, 8, 12**].

All sufficient conditions for STLC known today, and also the *Pontriagin Maximum Principle* and the *High Order Maximum Principle* [**6**], have in common that their proofs crucially rely on continuously parametrized families of piecewise constant control variations $\{u_s\}_{s \geq 0}$ (in this case of the zero control $u_0 \equiv 0$) with a fixed number of jumps, the parameter s being closely related to the amplitude of the control variation u_s and/or the length of the time intervals on which it is different from the reference control.

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