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A NOTE ON THE LOCATION OF COMPLEX ZEROS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. For second-order equations, $w'' + A(z)w = 0$, where $A(z) = a_m z^m + \cdots$ is a polynomial of degree $m \geq 1$, there is a classical result (due jointly to E. Hille, R. Nevanlinna and H. Wittich [**10**, p. 282]) which determines the possible location of the zeros of any solution $f \not\equiv 0$. The theorem states that for any $\varepsilon > 0$, all but finitely many zeros of f lie in the union (for $j = 0, 1, \ldots, m+1$) of the ε-sectors, $|\arg z - \phi_j| < \varepsilon$, where $\phi_j = (2\pi j - c)/(m+2)$ for any choice of $c = \arg a_m$. (The rays $\arg z = \phi_j$ are called "critical rays".) In this paper, we determine the situation for higher-order equations.

$$(1) \qquad w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_1(z)w' + a_0(z)w = 0 \qquad (n \geq 2),$$

where the $a_j(z)$ are polynomials. As shown in Theorem 1 below (§3), an interesting feature of the higher-order case is that the Hille-Nevanlinna-Wittich property (i.e., the existence of finitely many critical rays around which the zeros of any solution $f \not\equiv 0$ must be concentrated) need not hold when $n > 2$. There are equations (e.g. see §4 below) which have the property that for any ray, and any ε-sector around it, some solution $f \not\equiv 0$ has infinitely many zeros in the ε-sector. In Theorem 1, we show that in general either this latter property or the Hille-Nevanlinna-Wittich property holds for a given equation (1), and one can easily determine from the equation which of the two holds. In §7, we consider the problem of explicitly determining the critical rays for those equations (1) possessing the Hille-Nevanlinna-Wittich property.

The key tools in the proof of Theorem 1 are asymptotic existence theorems which were proved in [**4**] and [**6**] using the Strodt theory [**8, 9**]. (Details of the proof will appear elsewhere.)

2. Preliminaries. Given an equation (1) where the $a_j(z)$ are any rational functions, we first rewrite the equation (1) in terms of the operator θ defined by $\theta w = zw'$. (It is easy to prove by induction that for each $m = 1, 2, \ldots,$

$$(2) \qquad w^{(m)} = z^{-m}\left(\sum_{j=1}^{m} b_{jm}\theta^j w\right),$$

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