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ARGUESIAN LATTICES WHICH ARE NOT LINEAR

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ABSTRACT. A *linear* lattice is one representable by commuting equivalence relations. We construct a sequence of finite lattices A_n ($n \geq 3$) with the properties: (i) A_n is not linear, (ii) every proper sublattice of A_n is linear, and (iii) any set of generators for A_n has at least n elements. In particular, A_n is then Arguesian for $n \geq 7$. This settles a question raised in 1953 by Jónsson.

1. Introduction. A lattice L is *linear* if it is representable by commuting equivalence relations. Jónsson [**6**] showed that any such lattice is Arguesian. Numerous equivalent forms of the Arguesian law are now known; it is a strong condition with important applications in coordinatization theory [**1**, **2**]. Nevertheless, the question raised by Jónsson, whether every Arguesian lattice is linear, has remained open until now.

Here we describe an infinite family $\{A_n\}$ ($n \geq 3$) of nonlinear lattices, Arguesian for $n \geq 7$ (and possibly for $n \geq 4$), settling Jónsson's question in the negative. Actually, we obtain more: a specific infinite sequence of identities strictly between Arguesian and linear, and a proof that the universal Horn theory of linear lattices is not finitely based.

2. The lattices A_n. Let $n \geq 3$. In what follows, all indices are modulo n, i.e., x_{i+1} means x_0 when $i = n-1$, etc. Let L_n be the lattice of all subspaces of a vector space v ($\dim v = 2n$) over a prime field $\mathbf{K}$ with at least 3 elements. Let $\{\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1}\}$ be a basis of v. Let

$$(1) \qquad m = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle, \qquad q_i = \langle \{\alpha_j | j \neq i\} \rangle, \qquad p_i = q_i \wedge q_{i+1}, \\ r_i = m \vee \langle \beta_i \rangle, \qquad s_i = r_{i-1} \vee r_i,$$

where $\langle \cdots \rangle$ denotes linear span. Let

$$(2) \qquad \tilde{A}_n = [0, m] \cup [m, v] \cup \bigcup_i [p_i, r_i] \cup \bigcup_i [q_i, s_i],$$

where $[x, y] = \{z | x \leq z \leq y\}$.

$\tilde{A}_n \subset L_n$ is a sublattice; the intervals in the union (2) are its maximal complemented intervals, or *blocks*; they are the blocks of a tolerance relation on $\tilde{A}_n$ [**5**]; as such, the set S of blocks acquires a lattice structure; specifically, $0_S = [0, m]$, $1_S = [m, v]$, $a_i = [p_i, r_i]$ are atoms, $b_i = [q_i, s_i]$ are coatoms, and $a_i < b_i, b_{i+1}$ defines the order relation.

Let $\overline{m}$ ($\dim \overline{m} = n$) be another vector space, with basis $\{\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1}\}$. Define $\overline{p}_i$, $\overline{q}_i$ by analogy with (1). Let $F = \bigcup_i [p_i, v]$; $F \subset \tilde{A}_n$ is an order filter. Within F, $\bigcup_i [p_i, m]$ is an order ideal. Set up a "twisting" isomorphism

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