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ALMOST COMMUTING MATRICES AND THE BROWN-DOUGLAS-FILLMORE THEOREM

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The purpose of this note is to announce a constructive proof of the following theorem of Brown, Douglas, and Fillmore [**1**] which yields a quantitative version subject to a certain natural resolvent condition. Complete proofs will appear elsewhere.

THEOREM 1 (BDF). *Let T be an operator on a Hilbert space $\mathcal{H}$ such that $T^*T - TT^*$ is compact, and such that the Fredholm index* $\operatorname{ind}(T-\lambda) = 0$ *whenever this is defined* ($\lambda \notin \sigma_e(T)$). *Then there is a compact operator K such that $T - K$ is normal.*

Our quantitative version yields an estimate of $\|K\|$ in terms of the homogeneous quantity $\|T^*T - TT^*\|^{1/2}$ provided the spectrum of T is in a natural quantitative sense close to the essential spectrum $\sigma_e(T)$. Indeed, if N is normal, and $\|T - N\| < \varepsilon$, then

$$\|(T - \lambda I)^{-1}\| < (\operatorname{dist}(\lambda, \sigma(N)) - \varepsilon)^{-1}.$$

So it is reasonable to assume this inequality when $\operatorname{dist}(\lambda, \sigma(N)) > \varepsilon$.

THEOREM 2. *Given a compact subset X of the plane, there is a continuous positive real-valued function f_X defined on* $[0, \infty)$ *such that* $f_X(0) = 0$ *with the following property.*

Let T be essentially normal and satisfy the BDF hypotheses:

(i) $\sigma_e(T) = X$,

(ii) $\operatorname{ind}(T - \lambda I) = 0$ *for all* $\lambda \notin X$.

Furthermore let T satisfy the quantitative hypotheses:

(iii) $\|T^*T - TT^*\|^{1/2} < \varepsilon$,

(iv) $\|(T - \lambda I)^{-1}\| < (\operatorname{dist}(\lambda, X) - \varepsilon)^{-1}$ *if* $\operatorname{dist}(\lambda, X) > \varepsilon$.

Then there is a compact operator K such that $\|K\| < f_X(\varepsilon)$ *and $T - K$ is a normal operator with spectrum X.*

The most important special case in our proof is the annulus $A = \{\lambda \in \mathbf{C}\colon R_1 \le |\lambda| \le R_2\}$ In this case, the result obtained is much stronger and applies to more general operators.

THEOREM 3. *Let T be an operator on a Hilbert space with* $\sigma_e(T) = A = \{\lambda \in \mathbf{C}\colon R_1 \le |\lambda| \le R_2\}$. *Suppose* $\|T\| = R_2$ *and* $\|T^{-1}\| = R_1^{-1}$. *Then there is an operator K such that*

$$\|K\| \le 104\|T^*T - TT^*\|^{1/2}$$

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