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NONCLASSICAL EIGENVALUE ASYMPTOTICS FOR OPERATORS OF SCHRÖDINGER TYPE

DAVID GURARIE

We consider operators in the form $A = -\nabla \cdot \rho\nabla + V(x)$ on $\mathbf{R}^n$, where metric $\rho = (\rho_{ij}(x)) \geq 0$ and potential $V(x) \geq 0$. The classical Weyl principle for asymptotic distribution of large eigenvalues of A states that the counting function

$$N(\lambda) = \#\{\lambda_j \leq \lambda\} \sim \operatorname{Vol}\{(x;\xi)|\ \rho\xi \cdot \xi + V(x) \leq \lambda\} \quad \text{as } \lambda \to \infty.$$

(See for instance [**Gu**].) Integrating out variable ξ we can rewrite it as

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int (\lambda - V)_+^{n/2} \frac{dx}{\sqrt{\det \rho}}. \tag{1}$$

If potential V and metric ρ are assumed to be homogeneous in x, $V(x) = |x|^\alpha V(x')$; $\rho_{ij}(x) = |x|^\beta \rho_{ij}(x')$, $x' = x/|x|$, then (1) reduces to

$$N(\lambda) \sim C\lambda^{[n/2+(1-\beta/2)n/\alpha]} \int V^{-(n/\alpha)(1-\beta/2)} \frac{dS}{\sqrt{\det \rho}}; \tag{2}$$

integration over the unit sphere S with constant

$$C = \frac{\omega_n}{(2\pi)^n \alpha} B\left(\frac{n}{2} + 1; \frac{n}{\alpha}(1 - \beta/2)\right),$$

which depends on the volume ω_n of the unit sphere in $\mathbf{R}^n$ and the beta function.

Assuming $\beta < 2$ we see that integral (2) becomes divergent if $V(x')$ vanishes to a sufficiently high order. The simplest such potential is $V(x,y) = |x|^\alpha |y|^\beta$ on $\mathbf{R}^n + \mathbf{R}^m$.

The Weyl (volume counting) principle, when applied to the corresponding Schrödinger operator $-\Delta + V(x)$, fails to predict discrete spectrum below any energy level $\lambda > 0$. However, as was shown by D. Robert [**Ro**] and B. Simon [**Si**], A has purely discrete spectrum $\{\lambda_j\} \to +\infty$ (for qualitative explanation of this phenomenon see [**Fe**]). Moreover, the "nonclassical" asymptotics of $N(\lambda)$ was derived for such A.

Recently M. Solomyak [**So**] studied a general class of Schrödinger operators $-\Delta + V(x)$ with homogeneous potentials V subject to the following constraint:

(A) zeros of V, $\{x\colon V(x) = 0\}$ form a smooth cone Σ in $\mathbf{R}^n$ of dimension m, and V vanishes on Σ "uniformly" to order $b > 0$.

Introducing variables $x \in \Sigma$ and $y \in N_x$ (the normal to Σ at $\{x\}$), hypothesis (A) means that there exists

$$\lim_{t\to 0} t^{-b} V(x + ty) = V_0(x,y).$$

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