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COUNTEREXAMPLES IN THE THEORY OF NONSELFADJOINT OPERATOR ALGEBRAS

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In this note we announce the answers to several questions which involve nonselfadjoint operator algebras. Detailed proofs will appear elsewhere.

We use the following notation. $\mathcal{H}$ is a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, and $\mathcal{B}_1(\mathcal{H})$ is the ideal of trace class operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $\{T\}'$ is the commutant of T and $\{T\}''$ is the double commutant of T.

$\mathcal{B}(\mathcal{H})$ is the dual of $\mathcal{B}_1(\mathcal{H})$ (see [**2**]) so that $\mathcal{B}(\mathcal{H})$ has a weak * topology. $\mathcal{A}(T)$ denotes the smallest weak * closed algebra containing T and I, while $\mathcal{W}(T)$ is the smallest weak operator closed algebra containing T and I. $\operatorname{Lat} T$ is the lattice of (closed) invariant subspaces of T, and $\operatorname{Alg}\operatorname{Lat} T = \{B \in \mathcal{B}(\mathcal{H})\colon \operatorname{Lat} T \subset \operatorname{Lat} B\}$. It is elementary that $\mathcal{A}(T) \subset \mathcal{W}(T) \subset \{T\}'' \subset \{T\}'$, that $\mathcal{W}(T) \subset \operatorname{Alg}\operatorname{Lat} T$, and that all of these sets except $\mathcal{A}(T)$ are weakly closed algebras. Further, T is said to be reflexive if $\mathcal{W}(T) = \operatorname{Alg}\operatorname{Lat} T$.

We will consider the following questions.

QUESTION 1. Does $\mathcal{W}(T) = \{T\}' \cap \operatorname{Alg}\operatorname{Lat} T$, $\forall T \in \mathcal{B}(\mathcal{H})$?

QUESTION 2. Does $\mathcal{W}(T) = \{T\}'' \cap \operatorname{Alg}\operatorname{Lat} T$, $\forall T \in \mathcal{B}(\mathcal{H})$?

QUESTION 3. Must $T^{(n)}$ be reflexive, $\forall T \in \mathcal{B}(\mathcal{H})$ and $\forall n > 1$? (Here $T^{(n)}$ denotes the direct sum of n copies of T.)

QUESTION 4. If T_1 and T_2 are reflexive operators, must $T_1 \oplus T_2$ be reflexive?

QUESTION 5. Does $\mathcal{A}(T) = \mathcal{W}(T)$, $\forall T \in \mathcal{B}(\mathcal{H})$?

QUESTION 6. Does $\mathcal{W}(T)$ have a separating vector, $\forall T \subset \mathcal{B}(\mathcal{H})$?

Before stating the last question, we need some additional notation. Since $\mathcal{W}(T)$ is weak * closed in $\mathcal{B}(\mathcal{H})$, $\mathcal{W}(T)$ is a dual space, with predual $\mathcal{W}(T)_* = \mathcal{B}_1(\mathcal{H})/\mathcal{W}(T)_\perp$. Here $\mathcal{W}(T)_\perp$ denotes the preannihilator of $\mathcal{W}(T)$. For each n, let $F_n \subset \mathcal{B}_1(\mathcal{H})$ denote the set of operators of rank $\leq n$.

QUESTION 7. Is $F_1/\mathcal{W}(T)_\perp$ dense in $\mathcal{W}(T)_*$, $\forall T \subset \mathcal{B}(\mathcal{H})$?

Some remarks regarding these questions are in order. There are some relations among the questions. For $n = 1, 2$, or 6, an affirmative answer to Question n implies an affirmative answer to Question $n+1$.

Question 1 was raised independently by D. Sarason and P. Rosenthal (see [**6**, p. 195] and [**7**]). Rosenthal also asked Question 2 in [**7**]. In [**4**], J. Deddens listed several open questions, including Questions 3 and 4, concerning reflexive operators.

Question 5 has been raised by many people. The question appears in [**2**]. In [**8**], D. Westwood gave an example of an operator T so that $\mathcal{A}(T) = \mathcal{W}(T)$ but so that the weak and weak * topologies are different on $\mathcal{A}(T)$.

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