BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 2, October 1986

POTENTIAL THEORY FOR THE SCHRÖDINGER EQUATION

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Recently there has been a wave of results [**2, 4, 5, 11, 15, 16, 17**], on what is now referred to as the conditional gauge theorem. These works were inspired by [**1** and **6**]. We prove this result in greater generality than before and derive interesting new consequences. Let

$$A = \sum \frac{\partial}{\partial x^j}\left(a_{ij}(x)\frac{\partial}{\partial x^i}\right)$$

be a uniformly elliptic operator whose coefficients are bounded measurable functions on a bounded Lipschitz domain $D \subseteq R^d$. Define the Kato class K_d as the class of functions on D such that

$$\lim_{\alpha\downarrow 0}\sup_{x\in D}\int_{|x-y|<\alpha}\frac{|V(y)|}{|x-y|^{d-2}}\,dy = 0.$$

Our approach is to prove results about the operator $L = A + V$ by using known results for A and studying the probabilistic quantity, the conditional gauge.

In order to introduce the conditional gauge let $p(t,x,y)$ be the Green function for the parabolic equation $A = \partial/\partial t$ on $D \times (0,\infty)$. Let (X_t, P_x) be the diffusion, killed at the exit time $\tau_D = \inf\{t > 0\colon X_t \in D\}$, whose transition density is $p(t,x,y)$. The analysis involves the diffusion X_t conditioned on its exit position. This conditioned diffusion, see [**10**], has transition density $p^z(t,x,y) = K_A(x,z)^{-1}p(t,x,y)K_A(y,z)$, where K_A is the kernel function for A on D, $x,y \in D$, $z \in \partial D$. We shall write $P_x^z(\cdot) = P_x(\cdot \mid X_{\tau_D} = z)$ and $e_V(t) = \exp\{\int_0^t V(x_s)\,ds\}$. The so-called gauge is the function on D, $F(1;x) \equiv E_x[e_V(\tau_D)]$ and the conditional gauge is defined on $D \times \partial D$ by $F(1;x,z) \equiv E_x^z[e_V(\tau_D)]$. Theorem 1 was first proven in [**12**] when $A = \Delta$, V is bounded and ∂D is C^2, later when $A = \Delta$, $V \in K_d$ and ∂D is $C^{1,1}$ in [**16**] and recently when $A = \Delta$, $V \in L^p$ for some $p > d/2$ and ∂D is Lipschitz in [**13**]. Our main result is the following.

THEOREM 1. *Suppose the uniformly elliptic*

$$A = \sum \frac{\partial}{\partial x^j}\left(a_{ij}(x)\frac{\partial}{\partial x^i}\right)$$

has bounded measurable coefficients, $V \in K_d$, and $D \subseteq R^d$ is bounded and Lipschitz. Then $F(1;x) < \infty$ for some $x \in D$ iff there is a positive constant c such that $c^{-1} \le F(1;x,z) \le c$, $(x,z) \in D \times \partial D$.

Received by the editors November 11, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60J45, 31B25.