# BOOK REVIEWS

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*Recurrence in ergodic theory and combinatorial number theory*, by H. Furstenberg, Princeton Univ. Press, 1981, vii + 199 pp., $25.00. ISBN 0-691-08269-3

The mathematical study of dynamical systems, comprising ergodic theory, topological dynamics, and differentiable dynamics (depending on whether the setting is a measure space, a topological space, or a differentiable manifold), arose from theoretical physics, especially Hamiltonian mechanics and statistical mechanics, and has drawn heavily from number theory, harmonic analysis, and differential geometry for both techniques and examples. Here are three important examples of applications of combinatorics or number theory to dynamical systems.

(1) Hadamard and Morse developed symbolic dynamics as a systematic way of making abstract dynamical systems susceptible to combinatorial analysis. If $X$ is the phase space, or set of possible states, of a dynamical system, and $T: X \to X$ is a transformation which "makes time go by", so that $T^n x$ is the state at time $n$ of a system which at time 0 is in state $x$, and if $\{X_0, X_1\}$ is a carefully chosen partition of $X$ into two disjoint sets, then the doubly infinite 0,1 sequence $\omega(x)$ for which $\omega(x)_k = i$ if and only if $T^k x \in X_i$ will capture much of the information in the entire trajectory (or orbit, or history) $\{T^n x: n \in \mathbf{Z}\}$ of $x$. In this way, one can use combinatorial properties of the result of the "coding" $x \to \omega(x)$ to study dynamically interesting properties of the system, such as the existence of periodic or recurrent orbits, the presence or absence of metric or topological transitivity or other mixing properties, and the entropy.

(2) The theorem of Kronecker that if $\alpha$ is irrational then $\{n\alpha: n \in \mathbf{Z}\}$ is dense mod 1 says, in dynamical terms, that an irrational rotation $R_\alpha$ of the circle is *minimal*: there are no proper closed invariant sets. Weyl's equidistribution theorem which states that $\{n\alpha: n \in \mathbf{Z}\}$ is equidistributed mod 1 (for each subinterval $I$ of the circle, $\operatorname{card}\{k: 1 \leqslant k \leqslant n, (k\alpha \bmod 1) \in I\}/n$ converges to the length of $I$) implies that an irrational rotation of the circle is *uniquely ergodic*: there is only one $R_\alpha$-invariant Borel probability measure on the circle (Lebesgue measure). Extensions and variations of these theorems have provided many nice examples in dynamical systems theory.

(3) In the "small denominators" problems of celestial mechanics, the existence of a solution of an equation connected with a dynamical system