BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 2, April 1986

IP_r-SETS, SZEMERÉDI'S THEOREM, AND RAMSEY THEORY

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We set forth here combinatorial theorems relating to Szemerédi's theorem on arithmetic progressions in sets of integers of positive density. The method of proof (which we shall not even outline here) rests heavily on ergodic theory, and is developed in detail in [**3**]. Here we shall focus on the combinatorial ramifications. The common thread in our results is that a subset of a group which is "substantial" in some quantitative sense must contain certain kinds of patterns or configurations. In describing these configurations a special role is played by what we call IP_r-sets. (The letters IP refer to idempotence of operators that occur in the ergodic-theoretic development of this subject.)

DEFINITION 1. Let H be an abelian group. An IP_r-set in H is a set of $2^r - 1$ elements $\{h_n\} \subset H$ indexed by subsets $\alpha \subset \{1, 2, \ldots, r\}$, $\alpha \neq \emptyset$ and satisfying: if $\alpha \cap \beta = \emptyset$, then $h_{\alpha\cup\beta} = h_\alpha + h_\beta$.

The configurations whose existence is obtained in our discussion come about as follows. We suppose $\{h_\alpha^{(1)}\}, \{h_\alpha^{(2)}\}, \ldots, \{h_\alpha^{(k)}\}$ are IP_r-sets in a group. For a "substantial" set of the type to be described, we will find an element x and an index $\alpha \subset \{1, 2, \ldots, r\}$ such that all the elements

$$x + h_\alpha^{(1)}, x + h_\alpha^{(2)}, \ldots, x + h_\alpha^{(k)}$$

are inside the given set. If, for example, $H = \mathbf{Z}$ and $h_\alpha^{(q)} = qh_\alpha$, where $\{h_\alpha\}$ is a given IP_r-set, then the above sequence has the form $x + h_\alpha$, $x + 2h_\alpha, \ldots, x + kh_\alpha$, i.e., forms an arithmetic progression.

Another possibility is to take $H = \mathbf{R}^d$ = d-dimensional Euclidean space and to let $v_1, v_2, \ldots, v_k$ be any k vectors in $\mathbf{R}^d$. If $\{\lambda_\alpha\}$ is an IP_r-set in $\mathbf{R}$ then $\{\lambda_\alpha v_1\}, \{\lambda_\alpha v_2\}, \ldots, \{\lambda_\alpha v_k\}$ is a family of IP_r-sets in $\mathbf{R}^d$. The sequence $x + \lambda_\alpha v_1, x + \lambda_\alpha v_2, \ldots, x + \lambda_\alpha v_k$ forms a configuration similar to the arbitrary preassigned finite configuration $\{v_1, \ldots, v_k\}$. Thus, a special case of our results will ensure the existence of the vertices of an equilateral triangle or a square in certain planar sets.

An example of a "substantial" subset of a group is a subset of positive density. We will generally be interested in finite sets and so a "substantial" subset can be described as follows. In any countable abelian group H there exists a sequence J_n of finite subsets such that for any element $h \in H$ the symmetric difference of J_n and $J_n + h$ tends to zero relative to the size of J_n ("Følner" sets). A substantial subset is a subset of a Følner set containing a proportion bounded away from zero of elements in the Følner set. We now formulate this precisely.

Received by the editors September 20, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05B30.