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GEL'FAND'S PROBLEM ON UNITARY REPRESENTATIONS ASSOCIATED WITH DISCRETE SUBGROUPS OF $\mathrm{PSL}_2(\mathbf{R})$

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In 1978 M.-F. Vignéras [**10**] gave a negative answer to the question posed by I. M. Gel'fand [**2**], who asked if the induced representation of $\mathrm{PSL}_2(\mathbf{R})$ on $L^2(\Gamma\backslash \mathrm{PSL}_2(\mathbf{R}))$ determines a discrete subgroup Γ up to conjugation. She constructed explicitly two nonconjugate discrete groups arising from indefinite quaternion algebras defined over number fields giving rise to isomorphic induced representations. Such examples are necessarily quite sporadic since there are only finitely many conjugacy classes of arithmetic groups with a fixed signature; see K. Takeuchi [**9**]. The purpose of this note is to give, in a rather simple way, a large family of (nonarithmetic) discrete groups that are not determined by their induced representations. The key idea is to reduce the problem to the case of finite groups where the situations are simple and well understood. We should point out that a similar idea can be applied to constructions of various isospectral Riemannian manifolds [**8**].

Our construction is based on the following proposition, which follows from standard facts about induced representations.

PROPOSITION. *Let G be a locally compact topological group, and let $\Gamma \subset \Gamma_1,\ \Gamma_2 \subset \Gamma_0$ be discrete subgroups, with Γ normal and of finite index in Γ_0. Then if the subgroups $\mathcal{H}_i = \Gamma_i/\Gamma,\ i = 1, 2,$ of $\mathcal{G} = \Gamma_0/\Gamma$ meet each conjugacy class of $\mathcal{G}$ in the same number of elements, the representations of G on the spaces $L^2(\Gamma\backslash G),\ i = 1, 2,$ are unitarily equivalent.*

To construct nonconjugate Γ_1 and Γ_2 in $\mathrm{PSL}_2(\mathbf{R})$, we first choose an appropriate triple $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ with the same induced representation $\mathrm{Ind}_{\mathcal{H}_1}^{\mathcal{G}}(1) \equiv \mathrm{Ind}_{\mathcal{H}_2}^{\mathcal{G}}(1)$. For instance, we let $\mathcal{G}$ be the semidirect product $(\mathbf{Z}/8\mathbf{Z})^\times \cdot (\mathbf{Z}/8\mathbf{Z})$ and set $\mathcal{H}_1 = \{(1,0),(3,0),(5,0),(7,0)\}$, $\mathcal{H}_2 = \{(1,0),(3,4),(5,4),(7,0)\}$. It is easy to check that $\mathcal{H}_1$ and $\mathcal{H}_2$ are not conjugate in $\mathcal{G}$, and each conjugacy class of $\mathcal{G}$ meets $\mathcal{H}_1$ and $\mathcal{H}_2$ in the same number of elements.

We then take a torsion-free discrete subgroup $\Gamma_0 \subset \mathrm{PSL}_2(\mathbf{R})$ satisfying the following conditions:

(i) the genus of the Riemann surface $\Gamma_0\backslash \mathrm{PSL}_2(\mathbf{R})/\mathrm{SO}(2)$ is greater than two;

(ii) Γ_0 is maximal in $\mathrm{PGL}_2(\mathbf{R})$; and

(iii) Γ_0 is nonarithmetic.

Since the fundamental group of a Riemann surface of genus k has a free group on k generators as quotient, we may always, if k is large enough, find a sur-

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