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SOME EXTREMAL FUNCTIONS IN FOURIER ANALYSIS

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1. Introduction. In the late 1930s A. Beurling observed that the entire function

$$(1.1)\quad B(z)=\left(\frac{\sin \pi z}{\pi}\right)^2\left\{\sum_{n=0}^{\infty}(z-n)^{-2}-\sum_{m=-\infty}^{-1}(z-m)^{-2}+2z^{-1}\right\}$$

satisfies a simple and useful extremal property. We have

$$(1.2)\qquad \operatorname{sgn}(x)\leqslant B(x)$$

for all real x and

$$(1.3)\qquad \int_{-\infty}^{\infty} B(x)-\operatorname{sgn}(x)\,dx=1.$$

The function $B(z)$ is entire of exponential type 2π, and Beurling showed that if $F(z)$ is any entire function of exponential type 2π satisfying $\operatorname{sgn}(x)\leqslant F(x)$ for all real x, then

$$(1.4)\qquad \int_{-\infty}^{\infty} F(x)-\operatorname{sgn}(x)\,dx\geqslant 1.$$

Moreover, he showed that there is equality in (1.4) if and only if $F(z)=B(z)$. As an application Beurling found an interesting inequality for almost periodic functions (we include it here in Theorem 15), but his results were never published.

In 1974 A. Selberg used the function $B(z)$ to obtain a sharp form of the large sieve inequality. Selberg noted that if $\chi_E(x)$ is the characteristic function of the interval $E=[\alpha,\beta]$ and

$$(1.5)\qquad C_E(z)=\frac{1}{2}\{B(\beta-z)+B(z-\alpha)\},$$

then

$$(1.6)\qquad \chi_E(x)\leqslant C_E(x)$$

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