BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

0025-5718/85 $1.00 + $.25 per page

Groups of divisibilty, by Jiří Močkoř, Mathematics and its Applications, D. Reidel Publishing Company, Dordrecht, 1983, 184 pp., $39.50. ISBN 9-0277-1539-4

Associated with any integral domain D there is a partially ordered group $G(D)$, called the group of divisibility of D. When D is a valuation domain, G is merely the value group; and in this case, ideal-theoretic properties of D are easily derived from corresponding properties of G, and conversely. Even in the general case, though, it has frequently proved useful to phrase a ring-theoretic problem in terms of the ordered group G, first solve the problem there, and then pull back the solution if possible to D. Lorenzen originally applied this technique to solve a problem of Krull, and Nakayama used it to produce counterexamples to another problem of Krull and to a related problem of Clifford. Thus, the basic idea is to use partially ordered groups to produce examples of domains, the advantage deriving from the fact that partially ordered groups abound, whereas domains are not so easy to come by.

The definition of $G(D)$ for a domain D with quotient field K runs as follows. A principal fractional ideal of D is a set of the form aD, $0 \neq a \in K$, and is denoted by (a). These ideals form a group $G(D)$ under the multiplication $(a)(b) = (ab)$. Sometimes the ideal (0) is thrown in and denoted by ∞. An equivalent way of thinking of $G(D)$ is to identify elements of K which differ by unit multiplies from D, i.e., $G(D) = K^*/U(D)$, where K^* denotes the multiplicative group of nonzero elements of K, and $U(D)$ denotes the multiplicative group of units in D. In spite of its multiplicative origin, G is usually written additively, probably because of a desire to picture it graphically; but in most treatments, and the present book is no exception, there is a certain amount of vacillation between multiplicative and additive notation. The partial ordering on G is defined by $(a) \geqslant (b)$ if $(a) \subset (b)$, i.e., by reverse inclusion (think of $(4) \geqslant (2)$ when D is the ring of integers).

For a simple example take D to be a UFD. Then $G(D)$ is the direct sum of I copies of Z, the additive group of integers, where the index set I has the same cardinality as the set of principal prime ideals of D and the ordering is coordinatewise. This is merely a reflection of the fact that for any nonzero element $a \in K$, the fractional ideal (a) may be uniquely written in the form $(a) = (p_1)^{i_1} \cdots (p_n)^{i_n}$, where $i_j \in Z$, and $p_1, \ldots, p_n$ are distinct irreducible elements of D. In particular, the G arising from a UFD is a lattice ordered group, so it is somewhat surprising to discover

THEOREM. *A* (*p.o. abelian*) *group G is lattice ordered* (*if and*) *only if it is the group of divisibility of a Bezout domain.*

A bezout domain is one for which finitely generated ideals are principal (thus, the noetherian Bezout domains are just the PIDs). The author calls this