BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985
© 1985 American Mathematical Society
0025-5718/85 $1.00 + $ 25 per page

Differential forms orthogonal to holomorphic functions or forms, and their properties, by L. A. Aizenberg and Sh. A. Dautov, Translations of Mathematical Monographs, Vol. 56, American Mathematical Society, Providence, Rhode Island, 1983, ix + 165 pp., $36.00. ISBN 0-8218-4508-X

A considerable portion of complex analysis in several variables is devoted to developing n-dimensional analogs of classical one-variable theorems, and much of its fascination derives from the fact that these analogs often take forms which are subtle and surprising at first glance and seem natural only with hindsight. As a simple example, consider the fact that the zero set $N(f) = \{z: f(z) = 0\}$ of a holomorphic function of one variable consists of isolated points. This result, as stated, is utterly false in higher dimensions, for the zeros of holomorphic function of more than one variable are *never* isolated. But one obtains a correct theorem in any number of dimensions by rephrasing the one-variable result suitably: namely, $N(f)$ is an analytic subvariety of complex codimension one. Another example comes from the theorems of Mittag-Leffler and Weierstrass on finding meromorphic or holomorphic functions with prescribed poles or zeros; to obtain their n-dimensional analogs, the so-called Cousin problems, one should reformulate them in terms of sheaf cohomology.

The book of Aizenberg and Dautov takes as its starting point the following characterization of boundary values of holomorphic functions of one variable, which is well known to the experts but perhaps not to the general public. Let D be a bounded open set in $\mathbf{C}$ with smooth boundary $\partial D = \overline{D} \setminus D$; let $A(D)$ be the set of continuous functions on $\overline{D}$ which are holomorphic on D, and $A(\partial D) = \{f|\partial D: f \in A(D)\}$.

THEOREM 1. *For a continuous function f on ∂D to be in $A(\partial D)$ it is necessary and sufficient that*

$$(1) \qquad \int_{\partial D} f(z)g(z)\,dz = 0 \quad \textit{for all } g \in A(\partial D).$$

The necessity is an immediate consequence of Cauchy's theorem. To prove the sufficiency, one defines a holomorphic function φ on $\mathbf{C} \setminus \partial D$ by plugging f into the Cauchy integral formula:

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w - z}\,dw.$$

It is not too hard to prove that the difference of the limits of $\varphi(z)$ as z approaches a point $z_0 \in \partial D$ along the normal to ∂D from the inside and from the outside equals $f(z_0)$. On the other hand, it follows from (1) that $\varphi(z) = 0$ for $z \notin \overline{D}$. One concludes that the function F on $\overline{D}$, defined by $F = \varphi$ on D and $F = f$ on ∂D, is continuous, so $f \in A(\partial D)$.