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BMO ESTIMATES AND RADIAL GROWTH OF BLOCH FUNCTIONS

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1. Introduction. A real harmonic function $u(z)$ in the open unit disk $\mathbf{D}$ belongs to the (real) Bloch space B if

$$\|u\|_B = \sup\{(1-|z|^2)|\operatorname{grad} u| : z \in \mathbf{D}\} < \infty. \tag{1}$$

The importance of the class B in both complex and harmonic analysis is highlighted by the following facts:

(a) If $f(z)$ is a univalent analytic function in $\mathbf{D}$, then $u(z) = \log|f'(z)| \in B$ and $\|u\|_B \leq 8$; conversely, if $u(z) \in B$ and $\|u\|_B \leq 1$, then there is a univalent analytic function $f(z)$ in D such that $u(z) = \log|f'(z)|$ (see [**5**, pp. 32, 172]).

(b) The functions $u \in B$ coincide (up to a constant) with the derivatives $(\partial/\partial x)U(re^{ix})$ of harmonic functions $U(z)$ such that $U(e^{ix})$ belongs to the Zygmund class Λ_* (see [**6**, Chapter 7, §3]).

Integrating (1) yields the following "trivial" estimate (we assume $u(0) = 0$):

$$|u(z)| \leq \frac{1}{2}\|u\|_B \log\frac{1+|z|}{1-|z|} \qquad (z \in \mathbf{D}). \tag{2}$$

However, for L_p norms of $u(z)$ on circles $|z| = r$ $(0 < r < 1)$ a much deeper estimate, due to J. G. Clunie and T. H. MacGregor [**1**], holds $(p > 0)$:

$$\|u(r,\cdot)\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |u(re^{ix})|^p\,dx\right)^{1/p} \leq A_p\|u\|_B\sqrt{|\log(1-r)|}, \tag{3}$$

where A_p are some constants (the case $p = 2$ was proved earlier by T. M. Flett [**2**, p. 71]). As shown in [**1**], (3) implies that

$$\lim_{r\to 1^-}\frac{u(re^{ix})}{|\log(1-r)|^\gamma} = 0 \tag{4}$$

for almost all x whenever $\gamma > 1/2$.

The need to better understand the nature of such "nontrivial" estimates as (3) and (4) is the main motivation of this paper. In this context the use of the BMO norm for estimating Bloch functions on the circles $|z| = r$ looks more promising, since both norms are invariant under Möbius shifts of the argument.

In what follows, a simple but sharp BMO estimate for Bloch functions is proved. This estimate, together with the John-Nirenberg theorem [**3**], is then used to obtain a stronger form of the asymptotic estimate (4).

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