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MODULAR FORMS ASSOCIATED TO CLOSED GEODESICS AND ARITHMETIC APPLICATIONS

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ABSTRACT. For any Fuchsian group of the first kind and any even weight greater than 2, we prove that the relative Poincaré series associated to closed geodesics generate the space of cusp forms of the given weight. Those series have very interesting geometrical and arithmetic properties. For arithmetic subgroups of $\mathrm{SL}_2\mathbf{R}$ (with or without cusps), our construction allows us to define two natural rational structures on the space of cusp forms.

Let Γ be a Fuchsian group of the first kind, acting on the upper-half plane $\mathcal{H}$, i.e. a discrete subgroup of $\mathrm{SL}_2\mathbf{R}$ with $\mathrm{vol}(\Gamma\setminus\mathcal{H})<\infty$. For each integer $k\geq 2$ the space of cusp forms of weight $2k$ on Γ, $S_{2k}(\Gamma)$, is a finite-dimensional complex Hilbert space with respect to the Petersson scalar product

$$\langle f,g\rangle=\iint_{\Gamma\setminus\mathcal{H}} f(z)\overline{g(z)}(\mathrm{Im}\ z)^{2k}\,dV,$$

where $dV=(dx\,dy)/y^2$ is the hyperbolic measure on $\mathcal{H}$ invariant with respect to the action of $\mathrm{SL}_2\mathbf{R}$.

Notations. $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$; $j(\gamma,z)=cz+d$; $f|_{2k}\gamma=f(\gamma z)j(\gamma,z)^{-2k}$.

The following proposition gives a general construction of cusp forms on Γ.

PROPOSITION 1. *Let Γ_0 be a subgroup of Γ and $f(z)$ a function holomorphic on $\mathcal{H}$ satisfying:*

(i) $f|_{2k}\gamma=f$ *for all* $\gamma\in\Gamma_0$.

(ii) $\iint_{\Gamma_0\setminus\mathcal{H}}|f(z)|(\mathrm{Im}\ z)^k\,dV<\infty$.

Then the relative Poincaré series

$$F(z)=\sum_{\gamma\in\Gamma_0\setminus\Gamma}(f|_{2k}\gamma)(z)$$

converges absolutely on $\mathcal{H}$, uniformly on compact sets, and $F(z)$ belongs to the space $S_{2k}(\Gamma)$.

To each hyperbolic element $\gamma_0=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ ($|\mathrm{tr}\ \gamma_0|>2$) we associate a quadratic function $Q_{\gamma_0}(z)=cz^2+(d-a)z-b$ which has its zeros in the two hyperbolic fixed points of γ_0 and satisfies $Q_{\gamma_0}(\gamma_0 z)=Q_{\gamma_0}(z)\cdot j^{-2}(\gamma_0,z)$. Applying Proposition 1 with $\Gamma_0=\langle\gamma_0\rangle$ and $f(z)=c/Q_{\gamma_0}(z)^k$, we obtain a relative Poincaré series, which we denote by θ_{k,γ_0}. Here c is an appropriate normalizing constant. We have $\theta_{k,\gamma_0}=\theta_{k,\gamma_1}$ if γ_0 and γ_1 are conjugate in Γ, i.e. if they define the same closed geodesics on $\Gamma\setminus\mathcal{H}$. For each $f\in S_{2k}(\Gamma)$ and

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